2017 China Northern Math Olympiad

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Grade 10

1 A sequence $\{a_n\}$ is defined as follows: $a_1 = 1, a_2 = \frac{1}{3}$, and for all $n \geq 1$, $\frac{(1+a_n)(1+a_{n+2})}{(1+a_n+1)^2} = \frac{a_n a_{n+2}}{a_{n+1}^2}$.

Prove that, for all $n \geq 1$, $a_1 + a_2 + ... + a_n < \frac{34}{21}$.

2 Prove that there exist infinitely many integers n which satisfy $2017^2 | 1^n + 2^n + ... + 2017^n$.

3 Let D be the midpoint of side BC of triangle ABC. Let E, F be points on sides AB, AC respectively such that $DE = DF$. Prove that $AE + AF = BE + CF \iff \angle EDF = \angle BAC$.

4 Let Q be a set of permutations of $1, 2, ..., 100$ such that for all $1 \leq a, b \leq 100$, a can be found to the left of b and adjacent to b in at most one permutation in Q. Find the largest possible number of elements in Q.

5 Triangle ABC has $AB > AC$ and $\angle A = 60^\circ$. Let M be the midpoint of BC, N be the point on segment AB such that $\angle BNM = 30^\circ$. Let D, E be points on AB, AC respectively. Let F, G, H be the midpoints of BE, CD, DE respectively. Let O be the circumcenter of triangle FGH. Prove that O lies on line MN.

6 Find all integers n such that there exists a concave pentagon which can be dissected into n congruent triangles.

7 Let $S(n)$ denote the sum of the digits of the base-10 representation of an natural number n. For example. $S(2017) = 2 + 0 + 1 + 7 = 10$. Prove that for all primes p, there exists infinitely many n which satisfy $S(n) \equiv n \mod p$.

8 Let $n > 1$ be an integer, and let $x_1, x_2, ..., x_n$ be real numbers satisfying $x_1, x_2, ..., x_n \in [0, n]$ with $x_1 x_2 ... x_n = (n - x_1)(n - x_2)...(n - x_n)$. Find the maximum value of $y = x_1 + x_2 + ... + x_n$.

Grade 11

1 Define sequence $(a_n) : a_1 = \mathrm{e}, a_2 = \mathrm{e}^3, \mathrm{e}^{1-k} a_n^{k+2} = a_{n+1} a_{n-1}^{2k}$ for all $n \geq 2$, where k is a positive real number. Find $\prod_{i=1}^{2017} a_i$.

2 Grade 10 P3

3 Grade 10 P4

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4 Positive intenger $n \geq 3$. $a_1, a_2, \cdots, a_n$ are n positive intengers that are pairwise coprime, satisfying that there exists $k_1, k_2, \cdots, k_n \in \{-1, 1\}, \sum_{i=1}^{n} k_i a_i = 0$. Are there positive intengers $b_1, b_2, \cdots, b_n$, for any $k \in \mathbb{Z}_+$, $b_1 + ka_1, b_2 + ka_2, \cdots, b_n + ka_n$ are pairwise coprime?

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5 Length of sides of regular hexagon $ABCDEF$ is a. Two moving points M, N moves on sides BC, DE, satisfy that $\angle MAN = \frac{\pi}{3}$. Prove that $AM \cdot AN - BM \cdot DN$ is a definite value.

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6 Define $S_r(n)$: digit sum of n in base r. For example, $38 = (1102)_3, S_3(38) = 1 + 1 + 0 + 2 = 4$.
Prove:
(a) For any $r > 2$, there exists prime p, for any positive intenger n, $S_r(n) \equiv n \mod p$.
(b) For any $r > 1$ and prime p, there exists infinitely many n, $S_r(n) \equiv n \mod p$.

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7 Grade 10 P8

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8 On Qingqing Grassland, there are 7 sheep numberd $1, 2, 3, 4, 5, 6, 7$ and 2017 wolves numberd $1, 2, \cdots, 2017$. We have such strange rules:
(1) Define $P(n)$: the number of prime numbers that are smaller than n. Only when $P(i) \equiv j \pmod 7$, wolf i may eat sheep j (he can also choose not to eat the sheep).
(2) If wolf i eat sheep j, he will immediately turn into sheep j.
(3) If a wolf can make sure not to be eaten, he really wants to experience life as a sheep.
Assume that all wolves are very smart, then how many wolves will remain in the end?
