Online Math Open Problems 2017

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– Spring

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1 Find the smallest positive integer that is relatively prime to each of $2, 20, 204,$ and 2048.

Proposed by Yannick Yao

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2 A positive integer n is called *bad* if it cannot be expressed as the product of two distinct positive integers greater than 1. Find the number of bad positive integers less than $100.$

Proposed by Michael Ren

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3 In rectangle $ABCD$, $AB = 6$ and $BC = 16$. Points P, Q are chosen on the interior of side AB such that $AP = PQ = QB$, and points R, S are chosen on the interior of side CD such that $CR = RS = SD$. Find the area of the region formed by the union of parallelograms $APCR$ and $QBSD$.

Proposed by Yannick Yao

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4 Lunasa, Merlin, and Lyrica each has an instrument. We know the following about the prices of their instruments:

(a) If we raise the price of Lunasa's violin by 50% and decrease the price of Merlin's trumpet by 50%, the violin will be $\$50$ more expensive than the trumpet;
(b) If we raise the price of Merlin's trumpet by 50% and decrease the price of Lyrica's piano by 50%, the trumpet will be $\$50$ more expensive than the piano.

Given these conditions only, there exist integers m and n such that if we raise the price of Lunasa's violin by $m\%$ and decrease the price of Lyrica's piano by $m\%$, the violin must be exactly $\$n$ more expensive than the piano. Find $100m+n$.

Proposed by Yannick Yao

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5 There are 15 (not necessarily distinct) integers chosen uniformly at random from the range from 0 to 999, inclusive. Yang then computes the sum of their units digits, while Michael computes the last three digits of their sum. The probability of them getting the same result is $\frac{m}{n}$ for relatively prime positive integers m, n. Find $100m+n$

Proposed by Yannick Yao

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6 Let $ABCDEF$ be a regular hexagon with side length 10 inscribed in a circle ω. X, Y, and Z are points on ω such that X is on minor arc AB, Y is on minor arc CD, and Z is on minor arc EF, where X may coincide with A or B (and similarly for Y and Z). Compute the square of the smallest possible area of XYZ.

Proposed by Michael Ren

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7 Let S be the set of all positive integers between 1 and 2017, inclusive. Suppose that the least common multiple of all elements in S is L. Find the number of elements in S that do not divide $\frac{L}{2016}$.

Proposed by Yannick Yao

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8 A five-digit positive integer is called [i]k-phobic[/i] if no matter how one chooses to alter at most four of the digits, the resulting number (after disregarding any leading zeroes) will not be a multiple of k. Find the smallest positive integer value of k such that there exists a k-phobic number.

Proposed by Yannick Yao

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9 Kevin is trying to solve an economics question which has six steps. At each step, he has a probability p of making a sign error. Let q be the probability that Kevin makes an even number of sign errors (thus answering the question correctly!). For how many values of $0 \le p \le 1$ is it true that $p + q = 1$?

Proposed by Evan Chen

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10 When Cirno walks into her perfect math class today, she sees a polynomial $P(x) = 1$ (of degree 0) on the blackboard. As her teacher explains, for her pop quiz today, she will have to perform one of the two actions every minute:

(a) Add a monomial to $P(x)$ so that the degree of P increases by 1 and P remains monic;
(b) Replace the current polynomial $P(x)$ by $P(x+1)$. For example, if the current polynomial is $x^2 + 2x + 3$, then she will change it to $(x+1)^2 + 2(x+1) + 3 = x^2 + 4x + 6$.

Her score for the pop quiz is the sum of coefficients of the polynomial at the end of 9 minutes. Given that Cirno (miraculously) doesn't make any mistakes in performing the actions, what is the maximum score that she can get?

Proposed by Yannick Yao

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11 Let a_1, a_2, a_3, a_4 be integers with distinct absolute values. In the coordinate plane, let $A_1 = (a_1, a_1^2)$, $A_2 = (a_2, a_2^2)$, $A_3 = (a_3, a_3^2)$ and $A_4 = (a_4, a_4^2)$. Assume that lines A_1A_2 and A_3A_4 intersect on the y-axis at an acute angle of θ. The maximum possible value for $\tan\theta$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

Proposed by James Lin

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12 Alice has an isosceles triangle M_0N_0P, where $M_0P = N_0P$ and $\angle M_0PN_0 = \alpha^\circ$. (The angle is measured in degrees.) Given a triangle M_iN_jP for nonnegative integers i and j, Alice may perform one of two *elongations*:

a) an M-*elongation*, where she extends ray $\overrightarrow{PM_i}$ to a point M_{i+1} where $M_iM_{i+1} = M_iN_j$ and removes the point M_i.
b) an N-*elongation*, where she extends ray $\overrightarrow{PN_j}$ to a point N_{j+1} where $N_jN_{j+1} = M_iN_j$ and removes the point N_j.

After a series of 5 elongations, k of which were M-elongations, Alice finds that triangle $M_kN_{5-k}P$ is an isosceles triangle. Given that 10α is an integer, compute 10α.

Proposed by Yannick Yao

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13 On a real number line, the points $1, 2, 3, \dots, 11$ are marked. A grasshopper starts at point 1, then jumps to each of the other 10 marked points in some order so that no point is visited twice, before returning to point 1. The maximal length that he could have jumped in total is L, and there are N possible ways to achieve this maximum. Compute $L + N$.

Proposed by Yannick Yao

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14 Let ABC be a triangle, not right-angled, with positive integer angle measures (in degrees) and circumcenter O. Say that a triangle ABC is *good* if the following three conditions hold:

(a) There exists a point $P \neq A$ on side AB such that the circumcircle of $\triangle POA$ is tangent to BO.
(b) There exists a point $Q \neq A$ on side AC such that the circumcircle of $\triangle QOA$ is tangent to CO.
(c) The perimeter of $\triangle APQ$ is at least $AB + AC$.

Determine the number of ordered triples $(\angle A, \angle B, \angle C)$ for which $\triangle ABC$ is good.

Proposed by Vincent Huang

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15 Let $\phi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Over all integers $1 \le n \le 100$, find the maximum value of $\phi(n^2 + 2n) - \phi(n^2)$.

Proposed by Vincent Huang

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16 Let S denote the set of subsets of $\{1, 2, \dots, 2017\}$. For two sets A and B of integers, define $A \circ B$ as the *symmetric difference* of A and B. (In other words, $A \circ B$ is the set of integers that are an element of exactly one of A and B.) Let N be the number of functions $f : S \to S$ such that $f(A \circ B) = f(A) \circ f(B)$ for all $A, B \in S$. Find the remainder when N is divided by 1000.

Proposed by Michael Ren

17 Let ABC be a triangle with $BC = 7, AB = 5$, and $AC = 8$. Let M, N be the midpoints of sides AC, AB respectively, and let O be the circumcenter of ABC. Let BO, CO meet AC, AB at P and Q, respectively. If MN meets PQ at R and OR meets BC at S, then the value of OS^2 can be written in the form $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $100m + n$.

Proposed by Vincent Huang

18 Let p be an odd prime number less than 10^5. Granite and Pomegranate play a game. First, Granite picks a integer $c \in \{2, 3, \dots, p-1\}$.
Pomegranate then picks two integers d and x, defines $f(t) = ct + d$, and writes x on a sheet of paper.
Next, Granite writes $f(x)$ on the paper, Pomegranate writes $f(f(x))$, Granite writes $f(f(f(x)))$, and so on, with the players taking turns writing.
The game ends when two numbers appear on the paper whose difference is a multiple of p, and the player who wrote the most recent number wins. Find the sum of all p for which Pomegranate has a winning strategy.

Proposed by Yang Liu

19 For each integer $1 \le j \le 2017$, let S_j denote the set of integers $0 \le i \le 2^{2017} - 1$ such that $\left\lfloor \frac{i}{2^{j-1}} \right\rfloor$ is an odd integer. Let P be a polynomial such that

$$P\left(x_0, x_1, \dots, x_{2^{2017}-1}\right) = \prod_{1 \le j \le 2017}\left(1 - \prod_{i \in S_j} x_i\right).$$

Compute the remainder when

$$\sum_{\left(x_0, \dots, x_{2^{2017}-1}\right) \in \{0,1\}^{2^{2017}}} P\left(x_0, \dots, x_{2^{2017}-1}\right)$$

is divided by 2017.

Proposed by Ashwin Sah

20 Let n be a fixed positive integer. For integer m satisfying $|m| \le n$, define $S_m = \displaystyle\sum_{\substack{i-j=m \\ 0 \le i,j \le n}} \frac{1}{2^{i+j}}$.

Then

$$\lim_{n \to \infty}\left(S_{-n}^2 + S_{-n+1}^2 + \dots + S_n^2\right)$$

can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p, q. Compute $100p + q$.

Proposed by Vincent Huang

21 Let $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function such that, for all $a, b \in \mathbb{Z}_{\geq 0}$:

$$f(a)^2 + f(b)^2 + f(a+b)^2 = 1 + 2f(a)f(b)f(a+b).$$

Furthermore, suppose there exists $n \in \mathbb{Z}_{\geq 0}$ such that $f(n) = 577$. Let S be the sum of all possible values of $f(2017)$. Find the remainder when S is divided by 2017.

Proposed by Zack Chroman

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22 Let $S = \{(x, y) \mid -1 \leq xy \leq 1\}$ be a subset of the real coordinate plane. If the smallest real number that is greater than or equal to the area of any triangle whose interior lies entirely in S is A, compute the greatest integer not exceeding $1000A$.

Proposed by Yannick Yao

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23 Determine the number of ordered quintuples (a, b, c, d, e) of integers with $0 \leq a < b < c < d < e \leq 30$ for which there exist polynomials $Q(x)$ and $R(x)$ with integer coefficients such that

$$x^a + x^b + x^c + x^d + x^e = Q(x)(x^5 + x^4 + x^2 + x + 1) + 2R(x).$$

Proposed by Michael Ren

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24 For any positive integer n, let S_n denote the set of positive integers which cannot be written in the form $an + 2017b$ for nonnegative integers a and b. Let A_n denote the average of the elements of S_n if the cardinality of S_n is positive and finite, and 0 otherwise. Compute

$$\left\lfloor \sum_{n=1}^{\infty} \frac{A_n}{2^n} \right\rfloor.$$

Proposed by Tristan Shin

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25 A *simple hyperplane* in $\mathbb{R}^4$ has the form

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$$

for some integers $k_1, k_2, k_3, k_4 \in \{-1, 0, 1\}$ that are not all zero. Find the number of regions that the set of all simple hyperplanes divide the unit ball $x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1$ into.

Proposed by Yannick Yao

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26 Let ABC be a triangle with $AB = 13, BC = 15, AC = 14$, circumcenter O, and orthocenter H, and let M, N be the midpoints of minor and major arcs BC on the circumcircle of ABC. Suppose $P \in AB, Q \in AC$ satisfy that P, O, Q are collinear and $PQ || AN$, and point I satisfies $IP \perp AB, IQ \perp AC$. Let H' be the reflection of H over line PQ, and suppose $H'I$ meets PQ

at a point T. If $\frac{MT}{NT}$ can be written in the form $\frac{\sqrt{m}}{n}$ for positive integers m, n where m is not divisible by the square of any prime, then find $100m + n$.

Proposed by Vincent Huang

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27 Let N be the number of functions $f : \mathbb{Z}/16\mathbb{Z} \to \mathbb{Z}/16\mathbb{Z}$ such that for all $a, b \in \mathbb{Z}/16\mathbb{Z}$:

$$f(a)^2 + f(b)^2 + f(a+b)^2 \equiv 1 + 2f(a)f(b)f(a+b) \pmod{16}.$$

Find the remainder when N is divided by 2017.

Proposed by Zack Chroman

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28 Let S denote the set of fractions $\dfrac{m}{n}$ for relatively prime positive integers m and n with $m+n \leq 10000$. The least fraction in S that is strictly greater than

$$\prod_{i=0}^{\infty}\left(1 - \frac{1}{10^{2i+1}}\right)$$

can be expressed in the form $\dfrac{p}{q}$, where p and q are relatively prime positive integers. Find $1000p + q$.

Proposed by James Lin

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29 Let ABC be a triangle with $AB = 2\sqrt{6}, BC = 5, CA = \sqrt{26}$, midpoint M of BC, circumcircle Ω, and orthocenter H. Let BH intersect AC at E and CH intersect AB at F. Let R be the midpoint of EF and let N be the midpoint of AH. Let AR intersect the circumcircle of AHM again at L. Let the circumcircle of ANL intersect Ω and the circumcircle of BNC at J and O, respectively. Let circles AHM and JMO intersect again at U, and let AU intersect the circumcircle of AHC again at $V \neq A$. The square of the length of CV can be expressed in the form $\dfrac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

Proposed by Michael Ren

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30 Let $p = 2017$ be a prime. Given a positive integer n, let T be the set of all $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$. A function $f : T \to \mathbb{Z}/p\mathbb{Z}$ is called an n-*determinant* if for every pair $1 \leq i, j \leq n$ with $i \neq j$,

$$f(A) = f(A'),$$

where A' is the matrix obtained by adding the jth row to the ith row.

Let a_n be the number of n-determinants. Over all $n \geq 1$, how many distinct remainders of a_n are possible when divided by $\dfrac{(p^p - 1)(p^{p-1} - 1)}{p - 1}$?

Proposed by Ashwin Sah

– Fall

1 Jingoistic James wants to teach his kindergarten class how to add in Chimese. The Chimese also use base-10 number system, but have replaced the digits 0-9 with ten of its own characters. For example, the two-digit number represents 16. What is the sum of the two-digit numbers and ?

[i]Proposed by James Lin

2 The numbers a, b, c, d are $1, 2, 2, 3$ in some order. What is the greatest possible value of $a^{b^{c^d}}$?

Proposed by Yannick Yao and James Lin

3 The USAMO is a 6 question test. For each question, you submit a positive integer number p of pages on which your solution is written. On the ith page of this question, you write the fraction i/p to denote that this is the ith page out of p for this question. When you turned in your submissions for the 2017 USAMO, the bored proctor computed the sum of the fractions for all of the pages which you turned in. Surprisingly, this number turned out to be 2017. How many pages did you turn in?

Proposed by Tristan Shin

4 Steven draws a line segment between every two of the points

$$A(2,2), B(-2,2), C(-2,-2), D(2,-2), E(1,0), F(0,1), G(-1,0), H(0,-1).$$

How many regions does he divide the square $ABCD$ into?

[i]Proposed by Michael Ren

5 Henry starts with a list of the first 1000 positive integers, and performs a series of steps on the list. At each step, he erases any nonpositive integers or any integers that have a repeated digit, and then decreases everything in the list by 1. How many steps does it take for Henry's list to be empty?

Proposed by Michael Ren

6 A convex equilateral pentagon with side length 2 has two right angles. The greatest possible area of the pentagon is $m+\sqrt{n}$, where m and n are positive integers. Find $100m+n$.

Proposed by Yannick Yao

7 Let S be a set of 13 distinct, pairwise relatively prime, positive integers. What is the smallest possible value of $\max_{s\in S} s - \min_{s\in S} s$?

[i]Proposed by James Lin

8 A permutation of $\{1, 2, 3, \dots, 16\}$ is called *blocksum-simple* if there exists an integer n such that the sum of any 4 consecutive numbers in the permutation is either n or $n+1$. How many blocksum-simple permutations are there?

Proposed by Yannick Yao

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9 Let a and b be positive integers such that $(2a+b)(2b+a) = 4752$. Find the value of ab.

Proposed by James Lin

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10 Determine the value of $-1+2+3+4-5-6-7-8-9+...+10000$, where the signs change after each perfect square.

[i]Proposed by Michael Ren

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11 Let $\{a, b, c, d, e, f, g, h, i\}$ be a permutation of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\gcd(c, d) = \gcd(f, g) = 1$ and

$$(10a+b)^{c/d} = e^{f/g}.$$

Given that $h > i$, evaluate $10h+i$.

Proposed by James Lin

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12 Bill draws two circles which intersect at X, Y. Let P be the intersection of the common tangents to the two circles and let Q be a point on the line segment connecting the centers of the two circles such that lines PX and QX are perpendicular. Given that the radii of the two circles are $3, 4$ and the distance between the centers of these two circles is 5, then the largest distance from Q to any point on either of the circles can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m+n$.

Proposed by Tristan Shin

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13 We define the sets of lattice points $S_0, S_1, \ldots$ as $S_0 = \{(0,0)\}$ and S_k consisting of all lattice points that are exactly one unit away from exactly one point in S_{k-1}. Determine the number of points in S_{2017}.

[i]Proposed by Michael Ren

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14 Let S be the set of all points $(x_1, x_2, x_3, \dots, x_{2017})$ in $\mathbb{R}^{2017}$ satisfying $|x_i| + |x_j| \le 1$ for any $1 \le i < j \le 2017$. The volume of S can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m+n$.

Proposed by Yannick Yao

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15 Find the number of integers $1 \le k \le 1336$ such that $\binom{1337}{k}$ divides $\binom{1337}{k-1}\binom{1337}{k+1}$.

Proposed by Tristan Shin

16 Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two parabolas with distinct directrices ℓ_1 and ℓ_2 and distinct foci F_1 and F_2 respectively. It is known that $F_1F_2||\ell_1||\ell_2$, F_1 lies on $\mathcal{P}_2$, and F_2 lies on $\mathcal{P}_1$. The two parabolas intersect at distinct points A and B. Given that $F_1F_2 = 1$, the value of AB^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

[i]Proposed by Yannick Yao

17 For a positive integer n, define $f(n) = \sum_{i=0}^{\infty} \frac{\gcd(i,n)}{2^i}$ and let $g : \mathbb{N} \to \mathbb{Q}$ be a function such that $\sum_{d|n} g(d) = f(n)$ for all positive integers n. Given that $g(12321) = \frac{p}{q}$ for relatively prime integers p and q, find $v_2(p)$.

Proposed by Michael Ren

18 Let a, b, c be real nonzero numbers such that $a + b + c = 12$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = 1.$$

Compute the largest possible value of $abc - (a + 2b - 3c)$.

Proposed by Tristan Shin

19 Tessa the hyper-ant is at the origin of the four-dimensional Euclidean space $\mathbb{R}^4$. For each step she moves to another lattice point that is 2 units away from the point she is currently on. How many ways can she return to the origin for the first time after exactly 6 steps?

[i]Proposed by Yannick Yao

20 Let $p = 2017$ be a prime. Suppose that the number of ways to place p indistinguishable red marbles, p indistinguishable green marbles, and p indistinguishable blue marbles around a circle such that no red marble is next to a green marble and no blue marble is next to a blue marble is N. (Rotations and reflections of the same configuration are considered distinct.) Given that $N = p^m \cdot n$, where m is a nonnegative integer and n is not divisible by p, and r is the remainder of n when divided by p, compute $pm + r$.

Proposed by Yannick Yao

21 Iris has an infinite chessboard, in which an 8×8 subboard is marked as Sacred. In order to preserve the Sanctity of this chessboard, her friend Rosabel wishes to place some indistinguishable Holy Knights on the chessboard (not necessarily within the Sacred subboard) such that:

- No two Holy Knights occupy the same square;
- Each Holy Knight attacks at least one Sacred square;
- Each Sacred square is attacked by exactly one Holy Knight.

In how many ways can Rosabel protect the Sanctity of Iris' chessboard? (A Holy Knight works in the same way as a knight piece in chess, that is, it attacks any square that is two squares away in one direction and one square away in a perpendicular direction. Note that a Holy Knight does *not* attack the square it is on.)

Proposed by Yannick Yao

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22 Given a sequence of positive integers $a_1, a_2, a_3, \dots, a_n$, define the *power tower function*

$$f(a_1, a_2, a_3, \dots, a_n) = a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_n}}}}}}.$$

Let $b_1, b_2, b_3, \dots, b_{2017}$ be positive integers such that for any i between 1 and 2017 inclusive,

$$f(a_1, a_2, a_3, \dots, a_i, \dots, a_{2017}) \equiv f(a_1, a_2, a_3, \dots, a_i + b_i, \dots, a_{2017}) \pmod{2017}$$

for all sequences $a_1, a_2, a_3, \dots, a_{2017}$ of positive integers greater than 2017. Find the smallest possible value of $b_1 + b_2 + b_3 + \dots + b_{2017}$.

[i]Proposed by Yannick Yao

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23 Call a nonempty set V of nonzero integers *victorious* if there exists a polynomial $P(x)$ with integer coefficients such that $P(0) = 330$ and that $P(v) = 2|v|$ holds for all elements $v \in V$. Find the number of victorious sets.

Proposed by Yannick Yao

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24 Senators Sernie Banders and Cedric "Ced" Truz of OMOrica are running for the office of Price Dent. The election works as follows: There are 66 states, each composed of many adults and 2017 children, with only the latter eligible to vote. On election day, the children each cast their vote with equal probability to Banders or Truz. A majority of votes in the state towards a candidate means they "win" the state, and the candidate with the majority of won states becomes the new Price Dent. Should both candidates win an equal number of states, then whoever had the most votes cast for him wins.

Let the probability that Banders and Truz have an unresolvable election, i.e., that they tie on both the state count and the popular vote, be $\frac{p}{q}$ in lowest terms, and let m, n be the remainders when p, q, respectively, are divided by 1009. Find $m + n$.

Proposed by Ashwin Sah

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25 For an integer k let T_k denote the number of k-tuples of integers $(x_1, x_2, ... x_k)$ with $0 \le x_i < 73$ for each i, such that $73 | x_1^2 + x_2^2 + ... + x_k^2 - 1$. Compute the remainder when $T_1 + T_2 + ... + T_{2017}$ is divided by 2017.

[i]Proposed by Vincent Huang

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26 Define a sequence of polynomials $P_0, P_1, ...$ by the recurrence $P_0(x) = 1, P_1(x) = x, P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$. Let $S = \left|P'_{2017}\left(\frac{i}{2}\right)\right|$ and $T = \left|P'_{17}\left(\frac{i}{2}\right)\right|$, where i is the imaginary unit. Then $\frac{S}{T}$ is a rational number with fractional part $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute m.

Proposed by Tristan Shin

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27 For a graph G on n vertices, let $P_G(x)$ be the unique polynomial of degree at most n such that for each $i = 0, 1, 2, \dots, n$, $P_G(i)$ equals the number of ways to color the vertices of the graph G with i distinct colors such that no two vertices connected by an edge have the same color. For each integer $3 \le k \le 2017$, define a k-*tasty* graph to be a connected graph on 2017 vertices with 2017 edges and a cycle of length k. Let the *tastiness* of a k-tasty graph G be the number of coefficients in $P_G(x)$ that are odd integers, and let t be the minimal tastiness over all k-tasty graphs with $3 \le k \le 2017$. Determine the sum of all integers b between 3 and 2017 inclusive for which there exists a b-tasty graph with tastiness t.

Proposed by Vincent Huang

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28 Let ABC be a triangle with $AB = 7, AC = 9, BC = 10$, circumcenter O, circumradius R, and circumcircle ω. Let the tangents to ω at B, C meet at X. A variable line ℓ passes through O. Let A_1 be the projection of X onto ℓ and A_2 be the reflection of A_1 over O. Suppose that there exist two points Y, Z on ℓ such that $\angle YAB + \angle YBC + \angle YCA = \angle ZAB + \angle ZBC + \angle ZCA = 90^\circ$, where all angles are directed, and furthermore that O lies inside segment YZ with $OY * OZ = R^2$. Then there are several possible values for the sine of the angle at which the angle bisector of $\angle AA_2O$ meets BC. If the product of these values can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c with b squarefree and a, c coprime, determine $a + b + c$.

[i]Proposed by Vincent Huang

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29 Let $p = 2017$. If A is an $n \times n$ matrix composed of residues $\pmod{p}$ such that $\det A \not\equiv 0 \pmod{p}$ then let ord(A) be the minimum integer $d > 0$ such that $A^d \equiv I \pmod{p}$, where I is the $n \times n$ identity matrix. Let the maximum such order be a_n for every positive integer n. Compute the sum of the digits when $\sum_{k=1}^{p+1} a_k$ is expressed in base p.

Proposed by Ashwin Sah

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30 We define the bulldozer of triangle ABC as the segment between points P and Q, distinct points in the plane of ABC such that $PA \cdot BC = PB \cdot CA = PC \cdot AB$ and $QA \cdot BC = QB \cdot CA = QC \cdot AB$. Let XY be a segment of unit length in a plane $\mathcal{P}$, and let $\mathcal{S}$ be the region of $\mathcal{P}$ that the bulldozer of XYZ sweeps through as Z varies across the points in $\mathcal{P}$ satisfying $XZ = 2YZ$. Find the greatest integer that is less than 100 times the area of $\mathcal{S}$.

Proposed by Michael Ren
