South Africa National Olympiad 2017

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by DylanN

1 Together, the two positive integers a and b have 9 digits and contain each of the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once. For which possible values of a and b is the fraction a/b closest to 1?

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2 Let $ABCD$ be a rectangle with side lengths $AB = CD = 5$ and $BC = AD = 10$. W, X, Y, Z are points on AB, BC, CD and DA respectively chosen in such a way that $WXYZ$ is a kite, where $\angle ZWX$ is a right angle. Given that $WX = WZ = \sqrt{13}$ and $XY = ZY$, determine the length of XY.

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3 A representation of $\frac{17}{20}$ as a sum of reciprocals

$$\frac{17}{20} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

is called a *calm representation* with k terms if the a_i are distinct positive integers and at most one of them is not a power of two.

(a) Find the smallest value of k for which $\frac{17}{20}$ has a calm representation with k terms.
(b) Prove that there are infinitely many calm representations of $\frac{17}{20}$.

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4 Andile and Zandre play a game on a 2017×2017 board. At the beginning, Andile declares some of the squares *forbidden*, meaning the nothing may be placed on such a square. After that, they take turns to place coins on the board, with Zandre placing the first coin. It is not allowed to place a coin on a forbidden square or in the same row or column where another coin has already been placed. The player who places the last coin wins the game.

What is the least number of squares Andile needs to declare as forbidden at the beginning to ensure a win? (Assume that both players use an optimal strategy.)

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5 Let ABC be a triangle with circumcircle Γ. Let D be a point on segment BC such that $\angle BAD = \angle DAC$, and let M and N be points on segments BD and CD, respectively, such that $\angle MAD = \angle DAN$. Let S, P and Q (all different from A) be the intersections of the rays AD, AM and AN with Γ, respectively.

Show that the intersection of SM and QD lies on Γ.

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6 Determine all pairs (P, d) of a polynomial P with integer coefficients and an integer d such that the equation $P(x) - P(y) = d$ has infinitely many solutions in integers x and y with $x \neq y$.

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