AIME Problems 1989

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1 Compute $\sqrt{(31)(30)(29)(28)+1}$.

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2 Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices?

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3 Suppose n is a positive integer and d is a single digit in base 10. Find n if

$$\frac{n}{810} = 0.d25d25d25\ldots$$

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4 If $a<b<c<d<e$ are consecutive positive integers such that $b+c+d$ is a perfect square and $a+b+c+d+e$ is a perfect cube, what is the smallest possible value of c?

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5 When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let $\frac{i}{j}$, in lowest terms, be the probability that the coin comes up heads in exactly 3 out of 5 flips. Find $i+j$.

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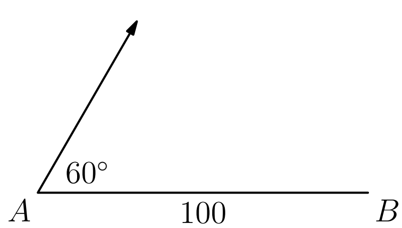
6 Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a 60° angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?



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7 If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic series. Find k.

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8 Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

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9 One of Euler's conjectures was disproved in then 1960s by three American mathematicians when they showed there was a positive integer n such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$

Find the value of n.

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10 Let a, b, c be the three sides of a triangle, and let α, β, γ, be the angles opposite them. If $a^2 + b^2 = 1989c^2$, find

$$\frac{\cot\gamma}{\cot\alpha + \cot\beta}.$$

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11 A sample of 121 integers is given, each between 1 and 1000 inclusive, with repetitions allowed. The sample has a unique mode (most frequent value). Let D be the difference between the mode and the arithmetic mean of the sample. What is the largest possible value of $\lfloor D \rfloor$? (For real x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

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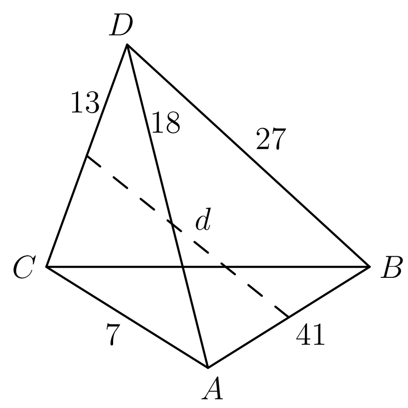
12 Let $ABCD$ be a tetrahedron with $AB = 41$, $AC = 7$, $AD = 18$, $BC = 36$, $BD = 27$, and $CD = 13$, as shown in the figure. Let d be the distance between the midpoints of edges AB and CD. Find d^2.



13 Let S be a subset of $\{1,2,3,\ldots,1989\}$ such that no two members of S differ by 4 or 7. What is the largest number of elements S can have?

14 Given a positive integer n, it can be shown that every complex number of the form $r+si$, where r and s are integers, can be uniquely expressed in the base $-n+i$ using the integers $1,2,\ldots,n^2$ as digits. That is, the equation

$$r+si=a_m(-n+i)^m+a_{m-1}(-n+i)^{m-1}+\cdots+a_1(-n+i)+a_0$$

is true for a unique choice of non-negative integer m and digits $a_0,a_1,\ldots,a_m$ chosen from the set $\{0,1,2,\ldots,n^2\}$, with $a_m\neq 0$. We write

$$r+si=(a_ma_{m-1}\ldots a_1a_0)_{-n+i}$$

to denote the base $-n+i$ expansion of $r+si$. There are only finitely many integers $k+0i$ that have four-digit expansions

$$k=(a_3a_2a_1a_0)_{-3+i}\quad a_3\neq 0.$$

Find the sum of all such k.

15 Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB (see the figure at right). Given that $AP=6$, $BP=9$, $PD=6$, $PE=3$, and $CF=20$, find the area of $\triangle ABC$.