South East Mathematical Olympiad 2017

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– Grade 10

Day 1 July 30th

1 Let $x_i \in \{0,1\}(i=1,2,\cdots,n)$. If the function $f=f(x_1,x_2,\cdots,x_n)$ only equals 0 or 1, then define f as an "n-variable Boolean function" and denote

$$D_n(f)=\{(x_1,x_2,\cdots,x_n)|f(x_1,x_2,\cdots,x_n)=0\}$$

. (1) Determine the number of n-variable Boolean functions; (2) Let g be a 10-variable Boolean function satisfying

$$g(x_1,x_2,\cdots,x_{10})\equiv 1+x_1+x_1x_2+x_1x_2x_3+\cdots+x_1x_2\cdots x_{10} \pmod 2$$

Evaluate the size of the set $D_{10}(g)$ and $\sum\limits_{(x_1,x_2,\cdots,x_{10})\in D_{10}(g)}(x_1+x_2+x_3+\cdots+x_{10})$.

2 Let ABC be an acute-angled triangle. In ABC, $AB\neq AB$, K is the midpoint of the the median AD, $DE\perp AB$ at E, $DF\perp AC$ at F. The lines KE, KF intersect the line BC at M, N, respectively. The circumcenters of $\triangle DEM$, $\triangle DFN$ are O_1,O_2, respectively.
Prove that $O_1O_2\parallel BC$.

3 For any positive integer n, let D_n denote the set of all positive divisors of n, and let $f_i(n)$ denote the size of the set

$$F_i(n)=\{a\in D_n | a\equiv i \pmod 4\}$$

where $i=1,2$.
Determine the smallest positive integer m such that $2f_1(m)-f_2(m)=2017$.

4 Let $a_1,a_2,\ldots,a_{2017}$ be reals satisfied $a_1=a_{2017}$, $|a_i+a_{i+2}-2a_{i+1}|\le 1$ for all $i=1,2,\ldots,2015$. Find the maximum value of $\max_{1\le i<j\le 2017}|a_i-a_j|$.

Day 2 July 31st

5 Let $ABCD$ be a cyclic quadrilateral inscribed in circle O, where $AC\perp BD$. M,N are the midpoint of arc ADC,ABC. DO and AN intersect each other at G, the line passes through G and parellel to NC intersect CD at K. Prove that $AK\perp BM$.

6 The sequence $\{a_n\}$ satisfies $a_1=\frac{1}{2}$, $a_2=\frac{3}{8}$, and $a_{n+1}^2+3a_na_{n+2}=2a_{n+1}(a_n+a_{n+2})(n\in\mathbb{N}^*)$. (1) Determine the general formula of the sequence $\{a_n\}$; (2) Prove that for any positive integer n, there is $0<a_n<\frac{1}{\sqrt{2n+1}}$.

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7 Let m be a given positive integer. Define $a_k=\frac{(2km)!}{3^{(k-1)m}}, k=1,2,\cdots$. Prove that there are infinite many integers and infinite many non-integers in the sequence $\{a_k\}$.

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8 Given the positive integer $m\geq 2$, $n\geq 3$. Define the following set

$$S=\{(a,b)|a\in\{1,2,\cdots,m\},b\in\{1,2,\cdots,n\}\}.$$

Let A be a subset of S. If there does not exist positive integers x_1,x_2,y_1,y_2,y_3 such that $x_1<x_2,y_1<y_2<y_3$ and

$$(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_2,y_2)\in A.$$

Determine the largest possible number of elements in A.

– Grade 11

Day 1 July 30th

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1 The same as Grade 10 Problem 2

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2 Let $x_i\in\{0,1\}(i=1,2,\cdots,n)$,if the value of function $f=f(x_1,x_2,\cdots,x_n)$ can only be 0 or 1,then we call f a n-var Boole function,and we denote $D_n(f)=\{(x_1,x_2,\cdots,x_n)|f(x_1,x_2,\cdots,x_n)=0\}$. (1) Find the number of n-var Boole function; (2) Let g be a n-var Boole function such that $g(x_1,x_2,\cdots,x_n)\equiv 1+x_1+x_1x_2+x_1x_2x_3+\cdots+x_1x_2\cdots x_n \pmod 2$,
Find the number of elements of the set $D_n(g)$,and find the maximum of $n\in\mathbb{N}_+$ such that $\sum_{(x_1,x_2,\cdots,x_n)\in D_n(g)}(x_1+x_2+\cdots+x_n)\leq 2017$.

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3 Let $a_1,a_2,\cdots,a_{n+1}>0$. Prove that

$$\sum_{i-1}^n a_i\sum_{i=1}^n a_{i+1}\geq\sum_{i=1}^n\frac{a_ia_{i+1}}{a_i+a_{i+1}}\cdot\sum_{i=1}^n(a_i+a_{i+1})$$

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4 For any positive integer n, let D_n denote the set of all positive divisors of n, and let $f_i(n)$ denote the size of the set

$$F_i(n)=\{a\in D_n|a\equiv i\pmod 4\}$$

where $i=0,1,2,3$.
Determine the smallest positive integer m such that $f_0(m)+f_1(m)-f_2(m)-f_3(m)=2017$.

Day 2 July 31st

5 Let a, b, c be real numbers, $a \neq 0$. If the equation $2ax^2 + bx + c = 0$ has real root on the interval $[-1, 1]$.
Prove that

$$\min\{c, a + c + 1\} \leq \max\{|b - a + 1|, |b + a - 1|\},$$

and determine the necessary and sufficient conditions of a, b, c for the equality case to be achieved.

6 Let $ABCD$ be a cyclic quadrilateral inscribed in circle O, where $AC \perp BD$. M be the midpoint of arc ADC. Circle (DOM) intersect DA, DC at E, F. Prove that $BE = BF$.

7 Find the maximum value of n, such that there exist n pairwise distinct positive numbers $x_1, x_2, \cdots, x_n$, satisfy

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 2017$$

8 Given the positive integer $m \geq 2, n \geq 3$. Define the following set

$$S = \{(a, b) | a \in \{1, 2, \cdots, m\}, b \in \{1, 2, \cdots, n\}\}.$$

Let A be a subset of S. If there does not exist positive integers $x_1, x_2, x_3, y_1, y_2, y_3$ such that $x_1 < x_2 < x_3, y_1 < y_2 < y_3$ and

$$(x_1, y_2), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_3, y_2) \in A.$$

Determine the largest possible number of elements in A.