IMC 2017

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–	**Day 1**
1	Determine all complex numbers λ for which there exists a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.
2	Let $f : \mathbb{R} \to (0, \infty)$ be a differentiabe function, and suppose that there exists a constant $L > 0$ such that $$\lvert f'(x) - f'(y)\rvert \le L\lvert x - y\rvert$$ for all x, y. Prove that $$(f'(x))^2 < 2Lf(x)$$ holds for all x.
3	For any positive integer m, denote by $P(m)$ the product of positive divisors of m (e.g $P(6) = 36$). For every positive integer n define the sequence $$a_1(n) = n, \qquad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \dots, 2016)$$ Determine whether for every set $S \subset \{1, 2, \dots, 2017\}$, there exists a positive integer n such that the following condition is satisfied: For every k with $1 \le k \le 2017$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.
4	There are n people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group S of people such that at least $\frac{n}{2017}$ persons in S have exactly two friends in S.
5	Let k and n be positive integers with $n \ge k^2 - 3k + 4$, and let $$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \dots + c_0$$ be a polynomial with complex coefficients such that $$c_0c_{n-2} = c_1c_{n-3} = \dots = c_{n-2}c_0 = 0$$ Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.
–	**Day 2**

6 Let $f:[0;+\infty)\to\mathbb{R}$ be a continuous function such that $\lim\limits_{x\to+\infty}f(x)=L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty}\int\limits_0^1 f(nx)\,\mathrm{d}x=L.$$

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7 Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer n, let

$$q_n(x)=(x+1)^n p(x)+x^n p(x+1).$$

Prove that there are only finitely many numbers n such that all roots of $q_n(x)$ are real.

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8 Define the sequence $A_1,A_2,\dots$ of matrices by the following recurrence:

$$A_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad A_{n+1}=\begin{pmatrix}A_n&I_{2^n}\\I_{2^n}&A_n\end{pmatrix}\quad(n=1,2,\dots)$$

where I_m is the $m\times m$ identity matrix.

Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0<\lambda_1<\ldots<\lambda_n$ with multiplicities $\binom{n}{0},\binom{n}{1},\dots,\binom{n}{n}$, respectively.

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9 Define the sequence $f_1,f_2,\dots:[0,1)\to\mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1=1;\qquad f'_{n+1}=f_nf_{n+1}\quad\text{on } (0,1),\quad\text{and}\quad f_{n+1}(0)=1.$$

Show that $\lim\limits_{n\to\infty}f_n(x)$ exists for every $x\in[0,1)$ and determine the limit function.

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10 Let K be an equilateral triangle in the plane. Prove that for every $p>0$ there exists an $\varepsilon>0$ with the following property: If n is a positive integer, and $T_1,\dots,T_n$ are non-overlapping triangles inside K such that each of them is homothetic to K with a negative ratio, and

$$\sum_{\ell=1}^{n}\mathrm{area}(T_\ell)>\mathrm{area}(K)-\varepsilon,$$

then

$$\sum_{\ell=1}^{n}\mathrm{perimeter}(T_\ell)>p.$$
