**Kosovo National Mathematical Olympiad 2017**

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| – | Grade 9 |
| 1 | 1. Find all primes of the form $n^3-1$ . |
| 2 | 2 .Solve the inequality $\|x-1\|-2\|x-4\|>3+2x$ |
| 3 | 3.<br>3 red birds for 4 days eat 36 grams of seed, 5 blue birds for 3 days eat 60 gram of seed.<br>For how many days could be feed 2 red birds and 4 blue birds with 88 gr seed? |
| 4 | 4. Find all triples of consecutive numbers ,whose sum of squares is equal to some fourdigit number with all four digits being equal. |
| 5 | 5.<br>Given the point T in rectangle ABCD, the distances from T to A,B,C is 15,20,25.<br>Find the distance from T to D. |
| – | Grade 11 |
| 1 | Find all ordered pairs $(a,b)$, of natural numbers, where $1<a,b\leq 100$, such that $\frac{1}{\log_{a}{10}}+\frac{1}{\log_{b}{10}}$ is a natural number. |
| 2 | Prove that for every positive real $a,b,c$ the inequality holds : $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1\geq\frac{2\sqrt{2}}{3}(\sqrt{\frac{a+b}{c}}+\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}})$<br>When does the equality hold? |
| 3 | $n$ teams participated in a basketball tournament. Each team has played with each team exactly one game. There was no tie. If in the end of the tournament the $i$-th team has $x_i$ wins and $y_i$ loses $(1 \leq i \leq n)$ prove that: $\sum_{i=1}^{n} {x_i}^2 = \sum_{i=1}^{n} {y_i}^2$ |
| 4 | Prove that : $\cos{36}-\sin{18}=\frac{1}{2}$ |
| 5 | A sphere with ray $R$ is cut by two parallel planes. such that the center of the sphere is outside the region determined by these planes. Let $S_1$ and $S_2$ be the areas of the intersections, and $d$ the distance between these planes. Find the area of the intersection of the sphere with the plane parallel with these two planes, with equal distance from them. |

 

– Grade 12

**1** The sequence $a_n$ $n \in \mathbb{N}$ is given in a recursive way with $a_1 = 1$, $a_n = \prod_{i=1}^{n-1} a_i + 1$, for all $n \geq 2$. Determine the least number $M$, such that $\sum_{n=1}^{m} \frac{1}{a_n} < M$ for all $m \in \mathbb{N}$

**2** Solve the system of equations $x + y + z = \pi \tan x \tan z = 2 \tan y \tan z = 18$

**3** Let $a \geq 2$ a fixed natural number, and let $a_n$ be the sequence $a_n = a^{a^{\cdot^{\cdot^{a}}}}$ (e.g $a_1 = a$, $a_2 = a^a$, etc.). Prove that $(a_{n+1} - a_n)|(a_{n+2} - a_{n+1})$ for every natural number $n$.

**4** Prove the identity $\sum_{k=2}^{n} k(k-1)\binom{n}{k} = \binom{n}{2}2^{n-1}$ for all $n = 2, 3, 4, ...$

**5** Lines determined by sides $AB$ and $CD$ of the convex quadrilateral $ABCD$ intersect at point $P$. Prove that $\alpha + \gamma = \beta + \delta$ if and only if $PA \cdot PB = PC \cdot PD$, where $\alpha, \beta, \gamma, \delta$ are the measures of the internal angles of vertices $A, B, C, D$ respectively.