BMO TST 2017

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1 Given n numbers different from 0, ($n \in \mathbb{N}$) which are arranged randomly. We do the following operation: Choose some consecutive numbers in the given order and change their sign (i.e. $x \to -x$). What is the minimum number of operations needed, in order to make all the numbers positive for any given initial configuration of the n numbers?

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2 Given a random positive integer N. Prove that there exist infinitely many positive integers M whose none of its digits is 0 and such that the sum of the digits of $N \cdot M$ is same as sum of digits M.

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3 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that : $f(x)f(y)f(z) = 9f(z + xyf(z))$, where x, y, z, are three positive real numbers.

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4 The incircle of $\triangle A_0B_0C_0$, meets legs B_0C_0, C_0A_0, A_0B_0, respectively on points A, B, C, and the incircle of $\triangle ABC$, with center I, meets legs BC, CA, AB, on points A_1, B_1, C_1, respectively. We write with $\sigma(ABC)$, and $\sigma(A_1B_1C_1)$ the areas of $\triangle ABC$, and $\triangle A_1B_1C_1$ respectively. Prove that if $\sigma(ABC) = 2\sigma(A_1B_1C_1)$, then lines AA_0, BB_0, CC_0 are concurrent.

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5 Given a set A which contains n elements. For any two distinct subsets A_1, A_2 of the given set A, we fix the number of elements of $A_1 \cap A_2$. Find the sum of all the numbers obtained in the described way.
