AIME Problems 2001

www.artofproblemsolving.com/community/c4898

by joml88, 4everwise, rrusczyk

–	I
–	March 27th
1	Find the sum of all positive two-digit integers that are divisible by each of their digits.
2	A finite set $\mathcal{S}$ of distinct real numbers has the following properties: the mean of $\mathcal{S} \cup \{1\}$ is 13 less than the mean of $\mathcal{S}$, and the mean of $\mathcal{S} \cup \{2001\}$ is 27 more than the mean of $\mathcal{S}$. Find the mean of $\mathcal{S}$.
3	Find the sum of the roots, real and non-real, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$, given that there are no multiple roots.
4	In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.
5	An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is $(0, 1)$, one altitude is contained in the y-axis, and the length of each side is $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $m + n$.
6	A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.
7	Triangle ABC has $AB = 21$, $AC = 22$, and $BC = 20$. Points D and E are located on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$ and contains the center of the inscribed circle of triangle ABC. Then $DE = m/n$, where m and n are relatively prime positive integers. Find $m + n$.
8	Call a positive integer N a *7-10 double* if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?
9	In triangle ABC, $AB = 13$, $BC = 15$ and $CA = 17$. Point D is on $\overline{AB}$, E is on $\overline{BC}$, and F is on $\overline{CA}$. Let $AD = p \cdot AB$, $BE = q \cdot BC$, and $CF = r \cdot CA$, where p, q, and r are positive and

satisfy $p+q+r=2/3$ and $p^2+q^2+r^2=2/5$. The ratio of the area of triangle DEF to the area of triangle ABC can be written in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

10 Let S be the set of points whose coordinates x, y, and z are integers that satisfy $0 \le x \le 2$, $0 \le y \le 3$, and $0 \le z \le 4$. Two distinct points are randomly chosen from S. The probability that the midpoint of the segment they determine also belongs to S is m/n, where m and n are relatively prime positive integers. Find $m+n$.

11 In a rectangular array of points, with 5 rows and N columns, the points are numbered consecutively from left to right beginning with the top row. Thus the top row is numbered 1 through N, the second row is numbered $N+1$ through $2N$, and so forth. Five points, P_1, P_2, P_3, P_4, and P_5, are selected so that each P_i is in row i. Let x_i be the number associated with P_i. Now renumber the array consecutively from top to bottom, beginning with the first column. Let y_i be the number associated with P_i after the renumbering. It is found that $x_1 = y_2$, $x_2 = y_1$, $x_3 = y_4$, $x_4 = y_5$, and $x_5 = y_3$. Find the smallest possible value of N.

12 A sphere is inscribed in the tetrahedron whose vertices are $A = (6,0,0), B = (0,4,0), C = (0,0,2)$, and $D = (0,0,0)$. The radius of the sphere is m/n, where m and n are relatively prime positive integers. Find $m+n$.

13 In a certain circle, the chord of a d-degree arc is 22 centimeters long, and the chord of a $2d$-degree arc is 20 centimeters longer than the chord of a $3d$-degree arc, where $d < 120$. The length of the chord of a $3d$-degree arc is $-m+\sqrt{n}$ centimeters, where m and n are positive integers. Find $m+n$.

14 A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

15 The numbers 1, 2, 3, 4, 5, 6, 7, and 8 are randomly written on the faces of a regular octahedron so that each face contains a different number. The probability that no two consecutive numbers, where 8 and 1 are considered to be consecutive, are written on faces that share an edge is m/n, where m and n are relatively prime positive integers. Find $m+n$.

– II

– April 10th

1 Let N be the largest positive integer with the following property: reading from left to right, each pair of consecutive digits of N forms a perfect square. What are the leftmost three digits of N?

2 Each of the 2001 students at a high school studies either Spanish or French, and some study both. The number who study Spanish is between 80 percent and 85 percent of the school population, and the number who study French is between 30 percent and 40 percent. Let m be the smallest number of students who could study both languages, and let M be the largest number of students who could study both languages. Find $M-m$.

3 Given that

$$x_1 = 211,$$
$$x_2 = 375,$$
$$x_3 = 420,$$
$$x_4 = 523, \text{ and}$$
$$x_n = x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4} \text{ when } n \geq 5,$$

find the value of $x_{531} + x_{753} + x_{975}$.

4 Let $R = (8,6)$. The lines whose equations are $8y = 15x$ and $10y = 3x$ contain points P and Q, respectively, such that R is the midpoint of $\overline{PQ}$. The length of PQ equals $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5 A set of positive numbers has the triangle property if it has three distinct elements that are the lengths of the sides of a triangle whose area is positive. Consider sets $\{4, 5, 6, \ldots, n\}$ of consecutive positive integers, all of whose ten-element subsets have the triangle property. What is the largest possible value of n?

6 Square $ABCD$ is inscribed in a circle. Square $EFGH$ has vertices E and F on $\overline{CD}$ and vertices G and H on the circle. The ratio of the area of square $EFGH$ to the area of square $ABCD$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers and $m < n$. Find $10n + m$.

7 Let $\triangle PQR$ be a right triangle with $PQ = 90$, $PR = 120$, and $QR = 150$. Let C_1 be the inscribed circle. Construct $\overline{ST}$ with S on $\overline{PR}$ and T on $\overline{QR}$, such that $\overline{ST}$ is perpendicular to $\overline{PR}$ and tangent to C_1. Construct $\overline{UV}$ with U on $\overline{PQ}$ and V on $\overline{QR}$ such that $\overline{UV}$ is perpendicular to $\overline{PQ}$ and tangent to C_1. Let C_2 be the inscribed circle of $\triangle RST$ and C_3 the inscribed circle of $\triangle QUV$. The distance between the centers of C_2 and C_3 can be written as $\sqrt{10n}$. What is n?

8 A certain function f has the properties that $f(3x) = 3f(x)$ for all positive real values of x, and that $f(x) = 1 - \mid x - 2 \mid$ for $1 \leq x \leq 3$. Find the smallest x for which $f(x) = f(2001)$.

9 Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. The probability of obtaining a grid that does not have a 2-by-2 red square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

---

10 How many positive integer multiples of 1001 can be expressed in the form $10^j - 10^i$, where i and j are integers and $0 \le i < j \le 99$?

---

11 Club Truncator is in a soccer league with six other teams, each of which it plays once. In any of its 6 matches, the probabilities that Club Truncator will win, lose, or tie are each $\frac{1}{3}$. The probability that Club Truncator will finish the season with more wins than losses is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

---

12 Given a triangle, its midpoint triangle is obtained by joining the midpoints of its sides. A sequence of polyhedra P_i is defined recursively as follows: P_0 is a regular tetrahedron whose volume is 1. To obtain P_{i+1}, replace the midpoint triangle of every face of P_i by an outward-pointing regular tetrahedron that has the midpoint triangle as a face. The volume of P_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

---

13 In quadrilateral $ABCD$, $\angle BAD \cong \angle ADC$ and $\angle ABD \cong \angle BCD$, $AB = 8$, $BD = 10$, and $BC = 6$. The length CD may be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

---

14 There are $2n$ complex numbers that satisfy both $z^{28} - z^8 - 1 = 0$ and $|z| = 1$. These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \le \theta_1 < \theta_2 < \cdots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \cdots + \theta_{2n}$.

---

15 Let $EFGH$, $EFDC$, and $EHBC$ be three adjacent square faces of a cube, for which $EC = 8$, and let A be the eighth vertex of the cube. Let I, J, and K, be the points on $\overline{EF}$, $\overline{EH}$, and $\overline{EC}$, respectively, so that $EI = EJ = EK = 2$. A solid S is obtained by drilling a tunnel through the cube. The sides of the tunnel are planes parallel to $\overline{AE}$, and containing the edges, $\overline{IJ}$, $\overline{JK}$, and $\overline{KI}$. The surface area of S, including the walls of the tunnel, is $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. Find $m+n+p$.
