Switzerland Team Selection Test 2015

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–	Day 1
1	What is the maximum number of 1 1 boxes that can be colored black in a n n chessboard so that any 2 2 square contains a maximum of 2 black boxes?
2	Let a, b, c be real numbers greater than or equal to 1. Prove that $$\min\left(\frac{10a^2-5a+1}{b^2-5b+10},\frac{10b^2-5b+1}{c^2-5c+10},\frac{10c^2-5c+1}{a^2-5a+10}\right)\leq abc.$$
3	Let ABC be a triangle with $AB>AC$. Let D be a point on AB such that $DB=DC$ and M the middle of AC. The parallel to BC passing through D intersects the line BM in K. Show that $\angle KCD=\angle DAC$.
–	Day 2
4	Find all relatively prime integers a,b such that $$a^2+a=b^3+b$$
5	Let ABC be a triangle. The points $K,L,$ and M lie on the segments $BC,CA,$ and $AB,$ respectively, such that the lines $AK,BL,$ and CM intersect in a common point. Prove that it is possible to choose two of the triangles $ALM,BMK,$ and CKL whose inradii sum up to at least the inradius of the triangle ABC. *Proposed by Estonia*
6	Find all polynomial function P of real coefficients such that for all $x\in\mathbb{R}$ $$P(x)P(x+1)=P(x^2+2)$$
–	Day 3

7 Find all finite and non-empty sets A of functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that for all $f_1, f_2 \in A$, there exists $g \in A$ such that for all $x, y \in \mathbb{R}$

$$f_1\left(f_2(y) - x\right) + 2x = g(x+y)$$

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8 Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

Proposed by Evan Chen

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9 Let $n \geq 2$ be a positive integer. At the center of a circular garden is a guard tower. On the outskirt of the garden there are n garden dwarfs regularly spaced. In the tower are attentive supervisors. Each supervisor controls a portion of the garden delimited by two dwarfs.
We say that the supervisor A controls the supervisor B if the region of B is contained in that of A.
Among the supervisors there are two groups: the apprentices and the teachers. Each apprentice is controlled by exactly one teachers, and controls no one, while the teachers are not controlled by anyone.
The entire garden has the following maintenance costs:
- One apprentice costs 1 gold per year.
- One teacher costs 2 gold per year.
- A garden dwarf costs 2 gold per year.
Show that the garden dwarfs cost at least as much as the supervisors.

– Day 4

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10 Let $ABCD$ be a parallelogram. Suppose that there exists a point P in the interior of the parallelogram which is on the perpendicular bisector of AB and such that $\angle PBA = \angle ADP$
Show that $\angle CPD = 2\angle BAP$

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11 In Thailand there are n cities. Each pair of cities is connected by a one-way street which can be borrowed, depending on its type, only by bike or by car. Show that there is a city from which you can reach any other city, either by bike or by car.
Remark : It is not necessary to use the same means of transport for each city

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12 Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.
