AIME Problems 2004

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–	I
–	March 23rd
1	The digits of a positive integer n are four consecutive integers in decreasing order when read from left to right. What is the sum of the possible remainders when n is divided by 37?
2	Set A consists of m consecutive integers whose sum is $2m$, and set B consists of $2m$ consecutive integers whose sum is m. The absolute value of the difference between the greatest element of A and the greatest element of B is 99. Find m.
3	A convex polyhedron P has 26 vertices, 60 edges, and 36 faces, 24 of which are triangular, and 12 of which are quadrilaterals. A space diagonal is a line segment connecting two non-adjacent vertices that do not belong to the same face. How many space diagonals does P have?
4	A square has sides of length 2. Set S is the set of all line segments that have length 2 and whose endpoints are on adjacent sides of the square. The midpoints of the line segments in set S enclose a region whose area to the nearest hundredth is k. Find $100k$.
5	Alpha and Beta both took part in a two-day problem-solving competition. At the end of the second day, each had attempted questions worth a total of 500 points. Alpha scored 160 points out of 300 points attempted on the first day, and scored 140 points out of 200 points attempted on the second day. Beta who did not attempt 300 points on the first day, had a positive integer score on each of the two days, and Beta's daily success rate (points scored divided by points attempted) on each day was less than Alpha's on that day. Alpha's two-day success ratio was 300/500 = 3/5. The largest possible two-day success ratio that Beta could achieve is m/n, where m and n are relatively prime positive integers. What is $m+n$?
6	An integer is called snakelike if its decimal representation $a_1a_2a_3\cdots a_k$ satisfies $a_i<a_{i+1}$ if i is odd and $a_i>a_{i+1}$ if i is even. How many snakelike integers between 1000 and 9999 have four distinct digits?
7	Let C be the coefficient of x^2 in the expansion of the product $$(1-x)(1+2x)(1-3x)\cdots(1+14x)(1-15x).$$ Find $\lvert C\rvert$.

8 Define a regular n-pointed star to be the union of n line segments $P_1P_2, P_2P_3, \ldots, P_nP_1$ such that
• the points $P_1, P_2, \ldots, P_n$ are coplanar and no three of them are collinear, • each of the n line segments intersects at least one of the other line segments at a point other than an endpoint, • all of the angles at $P_1, P_2, \ldots, P_n$ are congruent, • all of the n line segments $P_2P_3, \ldots, P_nP_1$ are congruent, and • the path $P_1P_2, P_2P_3, \ldots, P_nP_1$ turns counterclockwise at an angle of less than 180 degrees at each vertex.

There are no regular 3-pointed, 4-pointed, or 6-pointed stars. All regular 5-pointed stars are similar, but there are two non-similar regular 7-pointed stars. How many non-similar regular 1000-pointed stars are there?

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9 Let ABC be a triangle with sides 3, 4, and 5, and $DEFG$ be a 6-by-7 rectangle. A segment is drawn to divide triangle ABC into a triangle U_1 and a trapezoid V_1 and another segment is drawn to divide rectangle $DEFG$ into a triangle U_2 and a trapezoid V_2 such that U_1 is similar to U_2 and V_1 is similar to V_2. The minimum value of the area of U_1 can be written in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

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10 A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Given that the probability that the circle will not touch diagonal AC is m/n, where m and n are relatively prime positive integers. Find $m+n$.

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11 A solid in the shape of a right circular cone is 4 inches tall and its base has a 3-inch radius. The entire surface of the cone, including its base, is painted. A plane parallel to the base of the cone divides the cone into two solids, a smaller cone-shaped solid C and a frustum-shaped solid F, in such a way that the ratio between the areas of the painted surfaces of C and F and the ratio between the volumes of C and F are both equal to k. Given that $k = m/n$, where m and n are relatively prime positive integers, find $m+n$.

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12 Let S be the set of ordered pairs (x, y) such that $0 < x \le 1, 0 < y \le 1$, and $\left[\log_2\left(\frac{1}{x}\right)\right]$ and $\left[\log_5\left(\frac{1}{y}\right)\right]$ are both even. Given that the area of the graph of S is m/n, where m and n are relatively prime positive integers, find $m+n$. The notation $[z]$ denotes the greatest integer that is less than or equal to z.

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13 The polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$$

has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \le a_2 \le a_3 \le \cdots \le a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m+n$.

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14 A unicorn is tethered by a 20-foot silver rope to the base of a magician's cylindrical tower whose radius is 8 feet. The rope is attached to the tower at ground level and to the unicorn at a height of 4 feet. The unicorn has pulled the rope taut, the end of the rope is 4 feet from the nearest point on the tower, and the length of the rope that is touching the tower is $\frac{a-\sqrt{b}}{c}$ feet, where $a, b,$ and c are positive integers, and c is prime. Find $a+b+c$.

15 For all positive integers x, let

$$f(x)=\begin{cases}1 & \text{if } x=1\\ \frac{x}{10} & \text{if } x \text{ is divisible by 10}\\ x+1 & \text{otherwise}\end{cases}$$

and define a sequence as follows: $x_1=x$ and $x_{n+1}=f(x_n)$ for all positive integers n. Let $d(x)$ be the smallest n such that $x_n=1$. (For example, $d(100)=3$ and $d(87)=7$.) Let m be the number of positive integers x such that $d(x)=20$. Find the sum of the distinct prime factors of m.

– II

– April 6th

1 A chord of a circle is perpendicular to a radius at the midpoint of the radius. The ratio of the area of the larger of the two regions into which the chord divides the circle to the smaller can be expressed in the form $\frac{a\pi+b\sqrt{c}}{d\pi-e\sqrt{f}}$, where a, b, c, d, e, and f are positive integers, a and e are relatively prime, and neither c nor f is divisible by the square of any prime. Find the remainder when the product $abcdef$ is divided by 1000.

2 A jar has 10 red candies and 10 blue candies. Terry picks two candies at random, then Mary picks two of the remaining candies at random. Given that the probability that they get the same color combination, irrespective of order, is m/n, where m and n are relatively prime positive integers, find $m+n$.

3 A solid rectangular block is formed by gluing together N congruent 1-cm cubes face to face. When the block is viewed so that three of its faces are visible, exactly 231 of the 1-cm cubes cannot be seen. Find the smallest possible value of N.

4 How many positive integers less than 10,000 have at most two different digits?

5 In order to complete a large job, 1000 workers were hired, just enough to complete the job on schedule. All the workers stayed on the job while the first quarter of the work was done, so the first quarter of the work was completed on schedule. Then 100 workers were laid off, so the second quarter of the work was completed behind schedule. Then an additional 100 workers were laid off, so the third quarter of the work was completed still further behind schedule. Given that all workers work at the same rate, what is the minimum number of additional workers, beyond the 800 workers still on the job at the end of the third quarter, that must be hired after three-quarters of the work has been completed so that the entire project can be completed on schedule or before?

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6 Three clever monkeys divide a pile of bananas. The first monkey takes some bananas from the pile, keeps three-fourths of them, and divides the rest equally between the other two. The second monkey takes some bananas from the pile, keeps one-fourth of them, and divides the rest equally between the other two. The third monkey takes the remaining bananas from the pile, keeps one-twelfth of them, and divides the rest equally between the other two. Given that each monkey receives a whole number of bananas whenever the bananas are divided, and the numbers of bananas the first, second, and third monkeys have at the end of the process are in the ratio $3:2:1$, what is the least possible total for the number of bananas?

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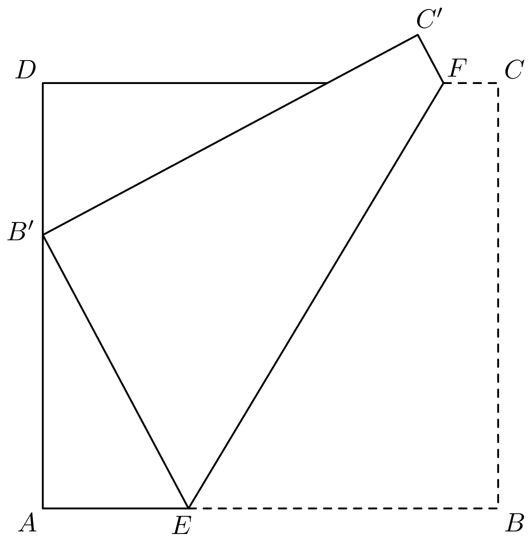
7 $ABCD$ is a rectangular sheet of paper that has been folded so that corner B is matched with point B' on edge AD. The crease is EF, where E is on AB and $F$is on CD. The dimensions $AE=8$, $BE=17$, and $CF=3$ are given. The perimeter of rectangle $ABCD$ is m/n, where m and n are relatively prime positive integers. Find $m+n$.



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8 How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

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9 A sequence of positive integers with $a_1=1$ and $a_9+a_{10}=646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n\ge 1$, the terms a_{2n-1}, a_{2n}, a_{2n+1} are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n+a_n$.

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10 Let S be the set of integers between 1 and 2^{40} whose binary expansions have exactly two 1's. If a number is chosen at random from S, the probability that it is divisible by 9 is p/q, where p and q are relatively prime positive integers. Find $p+q$.

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11 A right circular cone has a base with radius 600 and height $200\sqrt{7}$. A fly starts at a point on the surface of the cone whose distance from the vertex of the cone is 125, and crawls along the surface of the cone to a point on the exact opposite side of the cone whose distance from the vertex is $375\sqrt{2}$. Find the least distance that the fly could have crawled.

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12 Let $ABCD$ be an isosceles trapezoid, whose dimensions are $AB = 6$, $BC = 5 = DA$, and $CD = 4$. Draw circles of radius 3 centered at A and B, and circles of radius 2 centered at C and D. A circle contained within the trapezoid is tangent to all four of these circles. Its radius is $\frac{-k+m\sqrt{n}}{p}$, where k, m, n, and p are positive integers, n is not divisible by the square of any prime, and k and p are relatively prime. Find $k+m+n+p$.

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13 Let $ABCDE$ be a convex pentagon with $AB \parallel CE$, $BC \parallel AD$, $AC \parallel DE$, $\angle ABC = 120^\circ$, $AB = 3$, $BC = 5$, and $DE = 15$. Given that the ratio between the area of triangle ABC and the area of triangle EBD is m/n, where m and n are relatively prime positive integers, find $m+n$.

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14 Consider a string of n 7's, $7777\cdots77$, into which $+$ signs are inserted to produce an arithmetic expression. For example, $7+77+777+7+7=875$ could be obtained from eight 7's in this way. For how many values of n is it possible to insert $+$ signs so that the resulting expression has value 7000?

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15 A long thin strip of paper is 1024 units in length, 1 unit in width, and is divided into 1024 unit squares. The paper is folded in half repeatedly. For the first fold, the right end of the paper is folded over to coincide with and lie on top of the left end. The result is a 512 by 1 strip of double thickness. Next, the right end of this strip is folded over to coincide with and lie on top of the left end, resulting in a 256 by 1 strip of quadruple thickness. This process is repeated 8 more times. After the last fold, the strip has become a stack of 1024 unit squares. How many of these squares lie below the square that was originally the 942nd square counting from the left?
