AIME Problems 2007

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by mysmartmouth, bpms, Aegor, tjhance, worthawholebean, randomdragoon, nsato, mathfanatic, nat mc, probability1.01, MaThWhIz2004, Ignite168, pkothari13, rrusczyk

–	I
–	March 13th
1	How many positive perfect squares less than 10^6 are multiples of 24?
2	A 100 foot long moving walkway moves at a constant rate of 6 feet per second. Al steps onto the start of the walkway and stands. Bob steps onto the start of the walkway two seconds later and strolls forward along the walkway at a constant rate of 4 feet per second. Two seconds after that, Cy reaches the start of the walkway and walks briskly forward beside the walkway at a constant rate of 8 feet per second. At a certain time, one of these three persons is exactly halfway between the other two. At that time, find the distance in feet between the start of the walkway and the middle person.
3	The complex number z is equal to $9+bi$, where b is a positive real number and $i^2=-1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.
4	Three planets revolve about a star in coplanar circular orbits with the star at the center. All planets revolve in the same direction, each at a constant speed, and the periods of their orbits are 60, 84, and 140 years. The positions of the star and all three planets are currently collinear. They will next be collinear after n years. Find n.
5	The formula for converting a Fahrenheit temperature F to the corresponding Celsius temperature C is $C=\frac{5}{9}(F-32)$. An integer Fahrenheit temperature is converted to Celsius and rounded to the nearest integer; the resulting integer Celsius temperature is converted back to Fahrenheit and rounded to the nearest integer. For how many integer Fahrenheit temperatures T with $32 \leq T \leq 1000$ does the original temperature equal the final temperature?
6	A frog is placed at the origin on a number line, and moves according to the following rule: in a given move, the frog advanced to either the closest integer point with a greater integer coordinate that is a multiple of 3, or to the closest integer point with a greater integer coordinate that is a multiple of 13. A *move sequence* is a sequence of coordinates which correspond to valid moves, beginning with 0, and ending with 39. For example, 0, 3, 6, 13, 15, 26, 39 is a move sequence. How many move sequences are possible for the frog?

7 Let

$$N = \sum_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor).$$

Find the remainder when N is divided by 1000. (Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, and $\lceil x \rceil$ denotes the least integer that is greater than or equal to x.)

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8 The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 + (2k-43)x + k$ are both factors of $P(x)$?

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9 In right triangle ABC with right angle C, $CA = 30$ and $CB = 16$. Its legs $\overline{CA}$ and $\overline{CB}$ are extended beyond A and B. Points O_1 and O_2 lie in the exterior of the triangle and are the centers of two circles with equal radii. The circle with center O_1 is tangent to the hypotenuse and to the extension of leg CA, the circle with center O_2 is tangent to the hypotenuse and to the extension of leg CB, and the circles are externally tangent to each other. The length of the radius of either circle can be expressed as p/q, where p and q are relatively prime positive integers. Find $p+q$.

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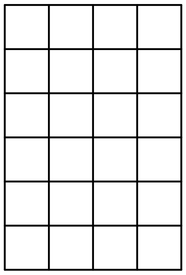
10 In the 6×4 grid shown, 12 of the 24 squares are to be shaded so that there are two shaded squares in each row and three shaded squares in each column. Let N be the number of shadings with this property. Find the remainder when N is divided by 1000.



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11 For each positive integer p, let $b(p)$ denote the unique positive integer k such that $|k - \sqrt{p}| < \frac{1}{2}$. For example, $b(6) = 2$ and $b(23) = 5$. If $S = \sum_{p=1}^{2007} b(p)$, find the remainder when S is divided by 1000.

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12 In isosceles triangle ABC, A is located at the origin and B is located at $(20, 0)$. Point C is in the first quadrant with $AC = BC$ and $\angle BAC = 75^\circ$. If $\triangle ABC$ is rotated counterclockwise about point A until the image of C lies on the positive y-axis, the area of the region common to the original and the rotated triangle is in the form $p\sqrt{2} + q\sqrt{3} + r\sqrt{6} + s$ where p, q, r, s are integers. Find $(p - q + r - s)/2$.

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13 A square pyramid with base $ABCD$ and vertex E has eight edges of length 4. A plane passes through the midpoints of $\overline{AE}$, $\overline{BC}$, and $\overline{CD}$. The plane's intersection with the pyramid has an area that can be expressed as $\sqrt{p}$. Find p.

14 Let a sequence be defined as follows: $a_1 = 3$, $a_2 = 3$, and for $n \geq 2$, $a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the largest integer less than or equal to $\frac{a_{2007}^2+a_{2006}^2}{a_{2007}a_{2006}}$.

15 Let ABC be an equilateral triangle, and let D and F be points on sides BC and AB, respectively, with $FA = 5$ and $CD = 2$. Point E lies on side CA such that $\angle DEF = 60^\circ$. The area of triangle DEF is $14\sqrt{3}$. The two possible values of the length of side AB are $p \pm q\sqrt{r}$, where p and q are rational, and r is an integer not divisible by the square of a prime. Find r.

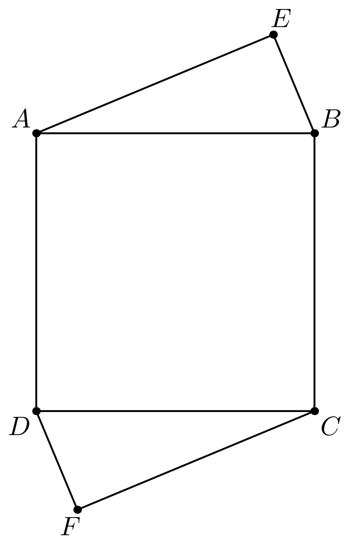
– II

– March 28th

1 A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in AIME and the four digits in 2007. No character may appear in a sequence more times than it appears among the four letters in AIME or the four digits in 2007. A set of plates in which each possible sequence appears exactly once contains N license plates. Find $\frac{N}{10}$.

2 Find the number of ordered triple (a, b, c) where a, b, and c are positive integers, a is a factor of b, a is a factor of c, and $a + b + c = 100$.

3 Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.



4 The workers in a factory produce widgets and whoosits. For each product, production time is constant and identical for all workers, but not necessarily equal for the two products. In one hour, 100 workers can produce 300 widgets and 200 whoosits. In two hours, 60 workers can produce 240 widgets and 300 whoosits. In three hours, 50 workers can produce 150 widgets and m whoosits. Find m.

5 The graph of the equation $9x + 223y = 2007$ is drawn on graph paper with each square representing one unit in each direction. How many of the 1 by 1 graph paper squares have interiors lying entirely below the graph and entirely in the first quadrant?

6 An integer is called *parity-monotonic* if its decimal representation $a_1a_2a_3\cdots a_k$ satisfies $a_i < a_{i+1}$ if a_i is odd, and $a_i > a_{i+1}$ is a_i is even. How many four-digit parity-monotonic integers are there?

7 Given a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For a certain integer k, there are exactly 70 positive integers $n_1, n_2, \ldots, n_{70}$ such that $k = \lfloor \sqrt[3]{n_1} \rfloor = \lfloor \sqrt[3]{n_2} \rfloor = \cdots = \lfloor \sqrt[3]{n_{70}} \rfloor$ and k divides n_i for all i such that $1 \le i \le 70$.
Find the maximum value of $\frac{n_i}{k}$ for $1 \le i \le 70$.

8 A rectangular piece of of paper measures 4 units by 5 units. Several lines are drawn parallel to the edges of the paper. A rectangle determined by the intersections of some of these lines is called *basic* if

(i) all four sides of the rectangle are segments of drawn line segments, and
(ii) no segments of drawn lines lie inside the rectangle.

Given that the total length of all lines drawn is exactly 2007 units, let N be the maximum possible number of basic rectangles determined. Find the remainder when N is divided by 1000.

9 Rectangle $ABCD$ is given with $AB = 63$ and $BC = 448$. Points E and F lie on AD and BC respectively, such that $AE = CF = 84$. The inscribed circle of triangle BEF is tangent to EF at point P, and the inscribed circle of triangle DEF is tangent to EF at point Q. Find PQ.

10 Let S be a set with six elements. Let P be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from P. the probability that B is contained in at least one of A or $S - A$ is $\frac{m}{n^r}$, where m, n, and r are positive integers, n is prime, and m and n are relatively prime. Find $m + n + r$. (The set $S - A$ is the set of all elements of S which are not in A.)

11 Two long cylindrical tubes of the same length but different diameters lie parallel to each other on a flat surface. The larger tube has radius 72 and rolls along the surface toward the smaller tube, which has radius 24. It rolls over the smaller tube and continues rolling along the flat surface until it comes to rest on the same point of its circumference as it started, having made one complete revolution. If the smaller tube never moves, and the rolling occurs with no slipping, the larger tube ends up a distance x from where it starts. The distance x can be expressed in the form $a\pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.

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12 The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that

$$\sum_{n=0}^{7} \log_3(x_n) = 308 \qquad \text{and} \qquad 56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57,$$

find $\log_3(x_{14})$.

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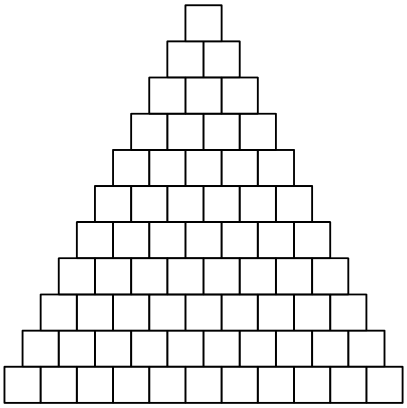
13 A triangular array of squares has one square in the first row, two in the second, and in general, k squares in the kth row for $1 \leq k \leq 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below (illustrated in given diagram). In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?



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14 Let $f(x)$ be a polynomial with real coefficients such that $f(0) = 1$, $f(2) + f(3) = 125$, and for all x, $f(x)f(2x^2) = f(2x^3 + x)$. Find $f(5)$.

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15 Four circles ω, ω_A, ω_B, and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B, and ω_C. If the sides of triangle ABC are 13, 14, and 15, the radius of ω can be represented in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find

$m+n$.

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