AIME Problems 2009

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–	I
–	March 18th
1	Call a 3-digit number *geometric* if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.
2	There is a complex number z with imaginary part 164 and a positive integer n such that $$\frac{z}{z+n}=4i.$$ Find n.
3	A coin that comes up heads with probability $p>0$ and tails with probability $1-p>0$ independently on each flip is flipped eight times. Suppose the probability of three heads and five tails is equal to $\frac{1}{25}$ of the probability of five heads and three tails. Let $p=\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.
4	In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB}=\frac{17}{1000}$ and point N is on $\overline{AD}$ so that $\frac{AN}{AD}=\frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.
5	Triangle ABC has $AC=450$ and $BC=300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK=CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM=180$, find LP.
6	How many positive integers N less than 1000 are there such that the equation $x^{\lfloor x\rfloor}=N$ has a solution for x? (The notation $\lfloor x\rfloor$ denotes the greatest integer that is less than or equal to x.)
7	The sequence (a_n) satisfies $a_1=1$ and $5^{(a_{n+1}-a_n)}-1=\dfrac{1}{n+\frac{2}{3}}$ for $n\geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.
8	Let $S=\{2^0,2^1,2^2,\ldots,2^{10}\}$. Consider all possible positive differences of pairs of elements of S. Let N be the sum of all of these differences. Find the remainder when N is divided by 1000.

9 A game show offers a contestant three prizes A, B and C, each of which is worth a whole number of dollars from $\$1$ to $\$9999$ inclusive. The contestant wins the prizes by correctly guessing the price of each prize in the order A, B, C. As a hint, the digits of the three prices are given. On a particular day, the digits given were $1, 1, 1, 1, 3, 3, 3$. Find the total number of possible guesses for all three prizes consistent with the hint.

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10 The Annual Interplanetary Mathematics Examination (AIME) is written by a committee of five Martians, five Venusians, and five Earthlings. At meetings, committee members sit at a round table with chairs numbered from 1 to 15 in clockwise order. Committee rules state that a Martian must occupy chair 1 and an Earthling must occupy chair 15. Furthermore, no Earthling can sit immediately to the left of a Martian, no Martian can sit immediately to the left of a Venusian, and no Venusian can sit immediately to the left of an Earthling. The number of possible seating arrangements for the committee is $N \cdot (5!)^3$. Find N.

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11 Consider the set of all triangles OPQ where O is the origin and P and Q are distinct points in the plane with nonnegative integer coordinates (x, y) such that $41x + y = 2009$. Find the number of such distinct triangles whose area is a positive integer.

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12 In right $\triangle ABC$ with hypotenuse $\overline{AB}$, $AC = 12$, $BC = 35$, and $\overline{CD}$ is the altitude to $\overline{AB}$. Let ω be the circle having $\overline{CD}$ as a diameter. Let I be a point outside $\triangle ABC$ such that $\overline{AI}$ and $\overline{BI}$ are both tangent to circle ω. The ratio of the perimeter of $\triangle ABI$ to the length AB can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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13 The terms of the sequence (a_i) defined by $a_{n+2} = \frac{a_n + 2009}{1 + a_{n+1}}$ for $n \geq 1$ are positive integers. Find the minimum possible value of $a_1 + a_2$.

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14 For $t = 1, 2, 3, 4$, define $S_t = \sum_{i=1}^{350} a_i^t$, where $a_i \in \{1, 2, 3, 4\}$. If $S_1 = 513$ and $S_4 = 4745$, find the minimum possible value for S_2.

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15 In triangle ABC, $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. The maximum possible area of $\triangle BPC$ can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

– II

– April 1st

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1 Before starting to paint, Bill had 130 ounces of blue paint, 164 ounces of red paint, and 188

ounces of white paint. Bill painted four equally sized stripes on a wall, making a blue stripe, a red stripe, a white stripe, and a pink stirpe. Pink is a mixture of red and white, not necessarily in equal amounts. When Bill finished, he had equal amounts of blue, red, and white paint left. Find the total number of ounces of paint Bill had left.

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2 Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

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3 In rectangle $ABCD$, $AB = 100$. Let E be the midpoint of $\overline{AD}$. Given that line AC and line BE are perpendicular, find the greatest integer less than AD.

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4 A group of children held a grape-eating contest. When the contest was over, the winner had eaten n grapes, and the child in kth place had eaten $n + 2 - 2k$ grapes. The total number of grapes eaten in the contest was 2009. Find the smallest possible value of n.

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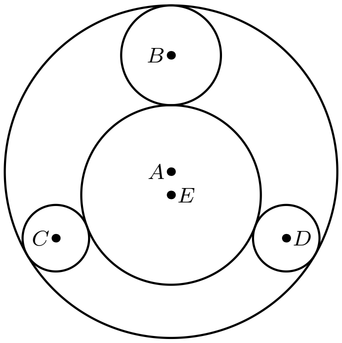
5 Equilateral triangle T is inscribed in circle A, which has radius 10. Circle B with radius 3 is internally tangent to circle A at one vertex of T. Circles C and D, both with radius 2, are internally tangent to circle A at the other two vertices of T. Circles B, C, and D are all externally tangent to circle E, which has radius $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



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6 Let m be the number of five-element subsets that can be chosen from the set of the first 14 natural numbers so that at least two of the five numbers are consecutive. Find the remainder when m is divided by 1000.

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7 Define $n!!$ to be $n(n-2)(n-4)\ldots 3\cdot 1$ for n odd and $n(n-2)(n-4)\ldots 4\cdot 2$ for n even. When $\sum_{i=1}^{2009} \frac{(2i-1)!!}{(2i)!!}$ is expressed as a fraction in lowest terms, its denominator is $2^a b$ with b odd. Find

$\frac{ab}{10}$.

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8 Dave rolls a fair six-sided die until a six appears for the first time. Independently, Linda rolls a fair six-sided die until a six appears for the first time. Let m and n be relatively prime positive integers such that $\frac{m}{n}$ is the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die. Find $m+n$.

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9 Let m be the number of solutions in positive integers to the equation $4x+3y+2z=2009$, and let n be the number of solutions in positive integers to the equation $4x+3y+2z=2000$. Find the remainder when $m-n$ is divided by 1000.

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10 Four lighthouses are located at points A, B, C, and D. The lighthouse at A is 5 kilometers from the lighthouse at B, the lighthouse at B is 12 kilometers from the lighthouse at C, and the lighthouse at A is 13 kilometers from the lighthouse at C. To an observer at A, the angle determined by the lights at B and D and the angle determined by the lights at C and D are equal. To an observer at C, the angle determined by the lights at A and B and the angle determined by the lights at D and B are equal. The number of kilometers from A to D is given by $\frac{p\sqrt{r}}{q}$, where p, q, and r are relatively prime positive integers, and r is not divisible by the square of any prime. Find $p+q+r$,

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11 For certain pairs (m,n) of positive integers with $m \geq n$ there are exactly 50 distinct positive integers k such that $|\log m - \log k| < \log n$. Find the sum of all possible values of the product mn.

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12 From the set of integers $\{1,2,3,\dots,2009\}$, choose k pairs $\{a_i,b_i\}$ with $a_i<b_i$ so that no two pairs have a common element. Suppose that all the sums a_i+b_i are distinct and less than or equal to 2009. Find the maximum possible value of k.

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13 Let A and B be the endpoints of a semicircular arc of radius 2. The arc is divided into seven congruent arcs by six equally spaced points $C_1,C_2,\dots,C_6$. All chords of the form $\overline{AC_i}$ or $\overline{BC_i}$ are drawn. Let n be the product of the lengths of these twelve chords. Find the remainder when n is divided by 1000.

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14 The sequence (a_n) satisfies $a_0=0$ and $a_{n+1}=\frac{8}{5}a_n+\frac{6}{5}\sqrt{4^n-a_n^2}$ for $n\geq 0$. Find the greatest integer less than or equal to a_{10}.

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15 Let $\overline{MN}$ be a diameter of a circle with diameter 1. Let A and B be points on one of the semicircular arcs determined by $\overline{MN}$ such that A is the midpoint of the semicircle and $MB=\frac{3}{5}$. Point C lies on the other semicircular arc. Let d be the length of the line segment whose endpoints are the intersections of diameter $\overline{MN}$ with the chords $\overline{AC}$ and $\overline{BC}$. The largest possible value of d can be written in the form $r - s\sqrt{t}$, where r, s, and t are positive integers and t is not divisible by the square of any prime. Find $r + s + t$.

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