India National Olympiad 1986

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by Leon

1 A person who left home between 4 p.m. and 5 p.m. returned between 5 p.m. and 6 p.m. and found that the hands of his watch had exactly exchanged place, when did he go out ?

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2 Solve

$$\begin{cases} \log_2 x + \log_4 y + \log_4 z = 2 \\ \log_3 y + \log_9 z + \log_9 x = 2 \\ \log_4 z + \log_{16} x + \log_{16} y = 2 \end{cases}$$

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3 Two circles with radii a and b respectively touch each other externally. Let c be the radius of a circle that touches these two circles as well as a common tangent to the two circles. Prove that

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

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4 Find the least natural number whose last digit is 7 such that it becomes 5 times larger when this last digit is carried to the beginning of the number.

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5 If $P(x)$ is a polynomial with integer coefficients and a, b, c, three distinct integers, then show that it is impossible to have $P(a) = b$, $P(b) = c$, $P(c) = a$.

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6 Construct a quadrilateral which is not a parallelogram, in which a pair of opposite angles and a pair of opposite sides are equal.

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7 If a, b, x, y are integers greater than 1 such that a and b have no common factor except 1 and $x^a = y^b$ show that $x = n^b$, $y = n^a$ for some integer n greater than 1.

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8 Suppose $A_1, \ldots, A_6$ are six sets each with four elements and $B_1, \ldots, B_n$ are n sets each with two elements, Let $S = A_1 \cup A_2 \cup \cdots \cup A_6 = B_1 \cup \cdots \cup B_n$. Given that each elements of S belogs to exactly four of the A's and to exactly three of the B's, find n.

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9 Show that among all quadrilaterals of a given perimeter the square has the largest area.
