India National Olympiad 1988

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by Leon

1 Let $m_1, m_2, m_3, \dots, m_n$ be a rearrangement of the numbers $1, 2, \dots, n$. Suppose that n is odd. Prove that the product

$$(m_1 - 1)(m_2 - 2) \cdots (m_n - n)$$

is an even integer.

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2 Prove that the product of 4 consecutive natural numbers cannot be a perfect cube.

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3 Five men, A, B, C, D, E are wearing caps of black or white colour without each knowing the colour of his cap. It is known that a man wearing black cap always speaks the truth while the ones wearing white always tell lies. If they make the following statements, find the colour worn by each of them: A : I see three black caps and one white cap. B : I see four white caps C : I see one black cap and three white caps D : I see your four black caps.

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4 If a and b are positive and $a + b = 1$, prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}$$

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5 Show that there do not exist any distinct natural numbers a, b, c, d such that $a^3 + b^3 = c^3 + d^3$ and $a + b = c + d$.

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6 If $a_0, a_1, \dots, a_{50}$ are the coefficients of the polynomial

$$\left(1 + x + x^2\right)^{25}$$

show that $a_0 + a_2 + a_4 + \cdots + a_{50}$ is even.

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7 Given an angle $\angle QBP$ and a point L outside the angle $\angle QBP$. Draw a straight line through L meeting BQ in A and BP in C such that the triangle $\triangle ABC$ has a given perimeter.

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8 A river flows between two houses A and B, the houses standing some distances away from the banks. Where should a bridge be built on the river so that a person going from A to B, using the bridge to cross the river may do so by the shortest path? Assume that the banks of the river are straight and parallel, and the bridge must be perpendicular to the banks.

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9 Show that for a triangle with radii of circumcircle and incircle equal to R, r respectively, the inequality $R \geq 2r$ holds.
