India National Olympiad 1989

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1 Prove that the Polynomial $f(x) = x^4 + 26x^3 + 56x^2 + 78x + 1989$ can't be expressed as a product $f(x) = p(x)q(x)$, where $p(x)$ and $q(x)$ are both polynomial with integral coefficients and with degree at least 1.

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2 Let a, b, c and d be any four real numbers, not all equal to zero. Prove that the roots of the polynomial $f(x) = x^6 + ax^3 + bx^2 + cx + d$ can't all be real.

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3 Let A denote a subset of the set $\{1, 11, 21, 31, \dots, 541, 551\}$ having the property that no two elements of A add up to 552. Prove that A can't have more than 28 elements.

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4 Determine all $n \in \mathbb{N}$ for which - n is not the square of any integer, - $\lfloor \sqrt{n} \rfloor^3$ divides n^2.

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5 For positive integers n, define $A(n)$ to be $\frac{(2n)!}{(n!)^2}$. Determine the sets of positive integers n for which

(a) $A(n)$ is an even number,

(b) $A(n)$ is a multiple of 4.

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6 Triangle ABC has incentre I and the incircle touches BC, CA at D, E respectively. Let BI meet DE at G. Show that AG is perpendicular to BG.

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7 Let A be one of the two points of intersection of two circles with centers X, Y respectively.The tangents at A to the two circles meet the circles again at B, C. Let a point P be located so that $PXAY$ is a parallelogram. Show that P is also the circumcenter of triangle ABC.
