India National Olympiad 1991

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1 Find the number of positive integers n for which

(i) $n \leq 1991$;

(ii) 6 is a factor of $(n^2 + 3n + 2)$.

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2 Given an acute-angled triangle ABC, let points A', B', C' be located as follows: A' is the point where altitude from A on BC meets the outwards-facing semicircle on BC as diameter. Points B', C' are located similarly.
Prove that $A[BCA']^2 + A[CAB']^2 + A[ABC']^2 = A[ABC]^2$ where $A[ABC]$ is the area of triangle ABC.

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3 Given a triangle ABC let

$$\begin{aligned} x &= \tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right) \\ y &= \tan\left(\frac{C-A}{2}\right)\tan\left(\frac{B}{2}\right) \\ z &= \tan\left(\frac{A-B}{2}\right)\tan\left(\frac{C}{2}\right). \end{aligned}$$

Prove that $x + y + z + xyz = 0$.

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4 Let a, b, c be real numbers with $0 < a < 1, 0 < b < 1, 0 < c < 1$, and $a + b + c = 2$.
Prove that $\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$.

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5 Triangle ABC has an incenter I. Let points X, Y be located on the line segments AB, AC respectively, so that $BX \cdot AB = IB^2$ and $CY \cdot AC = IC^2$. Given that the points X, I, Y lie on a straight line, find the possible values of the measure of angle A.

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6 (i) Determine the set of all positive integers n for which 3^{n+1} divides $2^{3^n} + 1$;

(ii) Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

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7 Solve the following system for real x, y, z

$$\{ \begin{array}{rcl} x+y-z & = & 4 \\ x^2-y^2+z^2 & = & -4 \\ xyz & = & 6. \end{array}$$

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8 There are 10 objects of total weight 20, each of the weights being a positive integers. Given that none of the weights exceeds 10 , prove that the ten objects can be divided into two groups that balance each other when placed on 2 pans of a balance.

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9 Triangle ABC has an incenter I I its incircle touches the side BC at T. The line through T parallel to IA meets the incircle again at S and the tangent to the incircle at S meets AB, AC at points C', B' respectively. Prove that triangle $AB'C'$ is similar to triangle ABC.

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10 For any positive integer n , let $s(n)$ denote the number of ordered pairs (x, y) of positive integers for which $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$. Determine the set of positive integers for which $s(n)=5$
