India National Olympiad 1992

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1 In a triangle ABC, $\angle A = 2 \cdot \angle B$. Prove that $a^2 = b(b+c)$.

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2 If $x, y, z \in \mathbb{R}$ such that $x+y+z=4$ and $x^2+y^2+z^2=6$, then show that each of x, y, z lies in the closed interval $\left[\frac{2}{3}, 2\right]$. Can x attain the extreme value $\frac{2}{3}$ or 2?

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3 Find the remainder when 19^{92} is divided by 92.

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4 Find the number of permutations $(p_1, p_2, p_3, p_4, p_5, p_6)$ of $1, 2, 3, 4, 5, 6$ such that for any $k, 1 \leq k \leq 5$, $(p_1, \ldots, p_k)$ does not form a permutation of $1, 2, \ldots, k$.

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5 Two circles C_1 and C_2 intersect at two distinct points P, Q in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B respectively. Let Y be the midpoint of AB and let QY meet the cirlces C_1 and C_2 in X and Z respectively. Show that Y is also the midpoint of XZ.

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6 Let $f(x)$ be a polynomial in x with integer coefficients and suppose that for five distinct integers $a_1, \ldots, a_5$ one has $f(a_1) = f(a_2) = \ldots = f(a_5) = 2$. Show that there does not exist an integer b such that $f(b) = 9$.

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7 Let $n \geq 3$ be an integer. Find the number of ways in which one can place the numbers $1, 2, 3, \ldots, n^2$ in the n^2 squares of a $n \times n$ chesboard, one on each, such that the numbers in each row and in each column are in arithmetic progression.

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8 Determine all pairs (m, n) of positive integers for which $2^m + 3^n$ is a perfect square.

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9 Let $A_1, A_2, \ldots, A_n$ be an n -sided regular polygon. If $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, find n.

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10 Determine all functions $f : \mathbb{R} - [0,1] \to \mathbb{R}$ such that

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}.$$
