Japan MO Finals 2017

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by maple116

1 Let a,b,c be positive integers. Prove that $lcm(a,b)\neq lcm(a+c,b+c)$.

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2 Let N be a positive integer. There are positive integers $a_1,a_2,\cdots,a_N$ and none of them are multiples of 2^{N+1}. For each integer $n\geq N+1$, set a_n as below:

If the remainder of a_k divided by 2^n is the smallest amongst the remainders of $a_1,\cdots,a_{n-1}$ divided by 2^n, set $a_n=2a_k$. If there are several integers k which satisfy the above condition, put the biggest one.

Prove the existence of a positive integer M which satisfies $a_n=a_M$ for $n\geq M$.

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3 Let ABC be an acute-angled triangle with the circumcenter O. Let D,E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A,Z,M are collinear.

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4 Let $n\geq 3$ be an integer. There are n people, and a meeting which at least 3 people attend is held everyday. Each attendant shake hands with the rest attendants at every meeting. After the nth meeting, every pair of the n people shook hands exactly once. Prove that every meeting was attended by the same number of attendants.

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5 Let $x_1,x_2,\cdots,x_{1000}$ be integers, and $\sum_{i=1}^{1000} x_i^k$ are all multiples of 2017 for any positive integers $k\leq 672$. Prove that $x_1,x_2,\cdots,x_{1000}$ are all multiples of 2017.
Note: 2017 is a prime number.
