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1 The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let O be the circumcenter of triangle APB and H be the orthocenter of triangle CPD. Show that the points H, P, O are collinear.

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2 Let $p(x) = x^2 + ax + b$ be a quadratic polynomial with $a, b \in \mathbb{Z}$. Given any integer n , show that there is an integer M such that $p(n)p(n+1) = p(M)$.

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3 If $a, b, c, d \in \mathbb{R}_+$ and $a + b + c + d = 1$, show that

$$ab + bc + cd \le \frac{1}{4}.$$

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4 Let ABC be a triangle in a plane π. Find the set of all points P (distinct from A, B, C) in the plane π such that the circumcircles of triangles ABP, BCP, CAP have the same radii.

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5 Show that there is a natural number n such that $n!$ when written in decimal notation ends exactly in 1993 zeros.

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6 Let ABC be a triangle right-angled at A and S be its circumcircle. Let S_1 be the circle touching the lines AB and AC, and the circle S internally. Further, let S_2 be the circle touching the lines AB and AC and the circle S externally. If r_1, r_2 be the radii of S_1, S_2 prove that $r_1 \cdot r_2 = 4A[ABC]$.

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7 Let $A = \{1, 2, 3, \ldots, 100\}$ and B be a subset of A having 53 elements. Show that B has 2 distinct elements x and y whose sum is divisible by 11.

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8 Let f be a bijective function from $A = \{1, 2, \ldots, n\}$ to itself. Show that there is a positive integer M such that $f^M(i) = f(i)$ for each i in A, where f^M denotes the composition $f \circ f \circ \cdots \circ f$ M times.

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9 Show that there exists a convex hexagon in the plane such that

(i) all its interior angles are equal;

(ii) its sides are $1, 2, 3, 4, 5, 6$ in some order.
