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by Rushil, yetti

1 In an acute angled triangle ABC, $\angle A = 30^\circ$, H is the orthocenter, and M is the midpoint of BC. On the line HM, take a point T such that $HM = MT$. Show that $AT = 2BC$.

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2 Show that there are infintely many pairs (a,b) of relatively prime integers (not necessarily positive) such that both the equations

$$\begin{aligned} x^2 + ax + b &= 0 \\ x^2 + 2ax + b &= 0 \end{aligned}$$

have integer roots.

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3 Show that the number of $3-$element subsets $\{a,b,c\}$ of $\{1,2,3,\ldots,63\}$ with $a+b+c < 95$ is less than the number of those with $a+b+c \geq 95$.

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4 Let ABC be a triangle and a circle Γ' be drawn lying outside the triangle, touching its incircle Γ externally, and also the two sides AB and AC. Show that the ratio of the radii of the circles Γ' and Γ is equal to $\tan^2\left(\dfrac{\pi - A}{4}\right)$.

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5 Let $n \geq 2$. Let $a_1, a_2, a_3, \ldots a_n$ be n real numbers all less than 1 and such that $|a_k - a_{k+1}| < 1$ for $1 \leq k \leq n-1$. Show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2n - 1.$$

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6 Find all primes p for which the quotient

$$\frac{2^{p-1} - 1}{p}$$

is a square.
