India National Olympiad 1996

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by Rushil

1 a) Given any positive integer n, show that there exist distint positive integers x and y such that $x+j$ divides $y+j$ for $j=1,2,3,\ldots,n$;

b) If for some positive integers x and y, $x+j$ divides $y+j$ for all positive integers j, prove that $x=y$.

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2 Let C_1 and C_2 be two concentric circles in the plane with radii R and $3R$ respectively. Show that the orthocenter of any triangle inscribed in circle C_1 lies in the interior of circle C_2. Conversely, show that every point in the interior of C_2 is the orthocenter of some triangle inscribed in C_1.

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3 Solve the following system for real a,b,c,d,e:

$$\left\{\begin{array}{ccc} 3a & = & (b+c+d)^3 \\ 3b & = & (c+d+e)^3 \\ 3c & = & (d+e+a)^3 \\ 3d & = & (e+a+b)^3 \\ 3e & = & (a+b+c)^3. \end{array}\right.$$

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4 Let X be a set containing n elements. Find the number of ordered triples (A,B,C) of subsets of X such that A is a subset of B and B is a proper subset of C.

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5 Define a sequence $(a_n)_{n\geq 1}$ by $a_1=1$ and $a_2=2$ and $a_{n+2}=2a_{n+1}-a_n+2$ for $n\geq 1$. prove that for any m , $a_m a_{m+1}$ is also a term in this sequence.

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6 There is a $2n\times 2n$ array (matrix) consisting of $0's$ and $1's$ and there are exactly $3n$ zeroes. Show that it is possible to remove all the zeroes by deleting some n rows and some n columns.
