India National Olympiad 1997

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by Rushil

1 Let $ABCD$ be a parallelogram. Suppose a line passing through C and lying outside the parallelogram meets AB and AD produced at E and F respectively. Show that

$$AC^2 + CE \cdot CF = AB \cdot AE + AD \cdot AF.$$

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2 Show that there do not exist positive integers m and n such that

$$\frac{m}{n} + \frac{n+1}{m} = 4.$$

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3 If a, b, c are three real numbers and

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a} = t$$

for some real number t, prove that $abc + t = 0$.

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4 In a unit square one hundred segments are drawn from the centre to the sides dividing the square into one hundred parts (triangles and possibly quadruilaterals). If all parts have equal perimetr p, show that $\frac{14}{10} < p < \frac{15}{10}$.

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5 Find the number of 4×4 array whose entries are from the set $\{0, 1, 2, 3\}$ and which are such that the sum of the numbers in each of the four rows and in each of the four columns is divisible by 4.

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6 Suppose a and b are two positive real numbers such that the roots of the cubic equation $x^3 - ax + b = 0$ are all real. If α is a root of this cubic with minimal absolute value, prove that

$$\frac{b}{a} < \alpha < \frac{3b}{2a}.$$
