India National Olympiad 1999

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1   Let ABC be an acute-angled triangle in which D, E, F are points on BC, CA, AB respectively such that $AD \perp BC$; $AE = BC$; and CF bisects $\angle C$ internally, Suppose CF meets AD and DE in M and N respectively. If $FM= 2, MN = 1, NC = 3$, find the perimeter of $\triangle ABC$.

2   In a village 1998 persons volunteered to clean up, for a fair, a rectangular field with integer sides and perimeter equla to 3996 feet. For this purpose, the field was divided into 1998 equal parts. If each part had an integer area, find the length and breadth of the field.

3   Show that there do not exist polynomials $p(x)$ and $q(x)$ each having integer coefficients and of degree greater than or equal to 1 such that

$$p(x)q(x) = x^5 + 2x + 1.$$

4   Let Γ and Γ' be two concentric circles. Let ABC and $A'B'C'$ be any two equilateral triangles inscribed in Γ and Γ' respectively. If P and P' are any two points on Γ and Γ' respectively, show that
$$P'A^2 + P'B^2 + P'C^2 = A'P^2 + B'P^2 + C'P^2.$$

5   Given any four distinct positive real numbers, show that one can choose three numbers A, B, C from among them, such that all three quadratic equations

$$
\begin{aligned}
Bx^2 + x + C &= 0 \\
Cx^2 + x + A &= 0 \\
Ax^2 + x + B &= 0
\end{aligned}
$$

have only real roots, or all three equations have only imaginary roots.

6   For which positive integer values of n can the set $\{1, 2, 3, \ldots, 4n\}$ be split into n disjoint 4-element subsets $\{a, b, c, d\}$ such that in each of these sets $a = \dfrac{b + c + d}{3}$.