India National Olympiad 2001

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1 Let ABC be a triangle in which no angle is 90°. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove that

(i) If P is the incenter or an excentre of ABC, then P is the circumenter of $A_1B_1C_1$;

(ii) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$;

(iii) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.

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2 Show that the equation $x^2 + y^2 + z^2 = (x-y)(y-z)(z-x)$ has infintely many solutions in integers x, y, z.

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3 If a, b, c are positive real numbers such that $abc = 1$, Prove that

$$a^{b+c}b^{c+a}c^{a+b} \leq 1.$$

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4 Show that given any nine integers, we can find four, a, b, c, d such that $a + b - c - d$is divisible by 20. Show that this is not always true for eight integers.

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5 ABC is a triangle. M is the midpoint of BC. $\angle MAB = \angle C$, and $\angle MAC = 15^\circ$. Show that $\angle AMC$ is obtuse. If O is the circumcenter of ADC, show that AOD is equilateral.

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6 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)f(xy)$ for all $x, y \in \mathbb{R}$.
