India National Olympiad 2002

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by Rushil, Fermat -Euler, seshadri

1 For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following statements.

(a_1) AB is parallel to DE. (a_2)$AE = BD$.

(b_1) BC is parallel to EF. (b_2)$BF = CE$.

(c_1) CD is parallel to FA. (c_2) $CA = DF$.

(a) Show that if all six of these statements are true then the hexagon is cyclic.

(b) Prove that, in fact, five of the six statements suffice.

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2 Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c . Find all a, b, c which give the smallest value

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3 If x, y are positive reals such that $x + y = 2$ show that $x^3y^3(x^3 + y^3) \leq 2$.

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4 Is it true that there exist 100 lines in the plane, no three concurrent, such that they intersect in exactly 2002 points?

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5 Do there exist distinct positive integers a, b, c such that a, b, c, $-a + b + c$, $a - b + c$, $a + b - c$, $a + b + c$ form an arithmetic progression (in some order).

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6 The numbers $1, 2, 3$, $\ldots$, n^2 are arranged in an $n \times n$ array, so that the numbers in each row increase from left to right, and the numbers in each column increase from top to bottom. Let a_{ij} be the number in position i, j. Let b_j be the number of possible values for a_{jj}. Show that

$$b_1 + b_2 + \cdots + b_n = \frac{n(n^2 - 3n + 5)}{3}.$$
