India National Olympiad 2007

www.artofproblemsolving.com/community/c4933

by Rijul saini, Sathej

1 In a triangle ABC right-angled at C , the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that
$$\frac{5}{2} < \frac{AB}{BC} < 3$$

---

2 Let n be a natural number such that $n = a^2 + b^2 + c^2$ for some natural numbers a, b, c. Prove that
$$9n = (p_1a + q_1b + r_1c)^2 + (p_2a + q_2b + r_2c)^2 + (p_3a + q_3b + r_3c)^2$$
where p_j's , q_j's , r_j's are all **nonzero** integers. Further, if 3 does **not** divide at least one of a, b, c, prove that $9n$ can be expressed in the form $x^2 + y^2 + z^2$, where x, y, z are natural numbers **none** of which is divisible by 3.

---

3 Let m and n be positive integers such that $x^2 - mx + n = 0$ has real roots α and β.

Prove that α and β are integers **if and only if** $[m\alpha] + [m\beta]$ is the square of an integer.

(Here $[x]$ denotes the largest integer not exceeding x)

---

4 Let $\sigma = (a_1, a_2, \cdots, a_n)$ be permutation of $(1, 2, \cdots, n)$. A pair (a_i, a_j) is said to correspond to an **inversion** of σ if $i < j$ but $a_i > a_j$. How many permutations of $(1, 2, \cdots, n)$, $n \geq 3$, have exactly **two** inversions?

For example, In the permutation $(2, 4, 5, 3, 1)$, there are 6 inversions corresponding to the pairs $(2, 1), (4, 3), (4, 1), (5, 3), (5, 1), (3, 1)$.

---

5 Let ABC be a triangle in which $AB = AC$. Let D be the midpoint of BC and P be a point on AD. Suppose E is the foot of perpendicular from P on AC. Define
$$\frac{AP}{PD} = \frac{BP}{PE} = \lambda, \quad \frac{BD}{AD} = m, \quad z = m^2(1 + \lambda)$$
Prove that
$$z^2 - (\lambda^3 - \lambda^2 - 2)z + 1 = 0$$

Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.

---

6 If x, y, z are positive real numbers, prove that

$$(x+y+z)^2\left(yz+zx+xy\right)^2\leq 3\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right).$$