India National Olympiad 2008

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1 Let ABC be triangle, I its in-center; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circum-circle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 are concylic, where I_1 is the in-center of triangle A_1, B_1, C_1.

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2 Find all triples (p, x, y) such that $p^x = y^4 + 4$, where p is a prime and x and y are natural numbers.

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3 Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that $a^2 + bc$ is a rational number for all distinct numbers a, b, c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A.

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4 All the points with integer coordinates in the xy-Plane are coloured using three colours, red, blue and green, each colour being used at least once. It is known that the point $(0, 0)$ is red and the point $(0, 1)$ is blue. Prove that there exist three points with integer coordinates of distinct colours which form the vertices of a right-angled triangle.

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5 Let ABC be a triangle; $\Gamma_A, \Gamma_B, \Gamma_C$ be three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB and BC; and Γ_C touches BC and CA. Let Γ be a circle touching circles $\Gamma_A, \Gamma_B, \Gamma_C$ externally. Prove that the line joining the circum-centre O and the in-centre I of triangle ABC passes through the centre of Γ.

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6 Let $P(x)$ be a polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that
(i) $P(x) \cdot Q(x)$ is a polynomial in x^2 , and
(ii) $P(x) \cdot R(x)$ is a polynomial in x^3.
