

**India National Olympiad 2009**

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by skand

**Day 1**

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**1** Let  $ABC$  be a triangle and let  $P$  be an interior point such that  $\angle BPC = 90^\circ$ ,  $\angle BAP = \angle BCP$ . Let  $M, N$  be the midpoints of  $AC, BC$  respectively. Suppose  $BP = 2PM$ . Prove that  $A, P, N$  are collinear.

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**2** Define a sequence  $\langle a_n \rangle_{n=1}^{\infty}$  as follows  
 $a_n = 0$ , if number of positive divisors of  $n$  is odd  
 $a_n = 1$ , if number of positive divisors of  $n$  is even

(The positive divisors of  $n$  include 1 as well as  $n$ .) Let  $x = 0.a_1a_2a_3\dots$  be the real number whose decimal expansion contains  $a_n$  in the  $n$ -th place,  $n \geq 1$ . Determine, with proof, whether  $x$  is rational or irrational.

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**3** Find all real numbers  $x$  such that:  $[x^2 + 2x] = [x]^2 + 2[x]$

(Here  $[x]$  denotes the largest integer not exceeding  $x$ .)

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**Day 2**

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**4** All the points in the plane are colored using three colors. Prove that there exists a triangle with vertices having the same color such that *either* it is isosceles *or* its angles are in geometric progression.

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**5** Let  $ABC$  be an acute angled triangle and let  $H$  be its orthocenter. Let  $h_{max}$  denote the largest altitude of the triangle  $ABC$ . Prove that:

$$AH + BH + CH \leq 2h_{max}$$

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**6** Let  $a, b, c$  be positive real numbers such that  $a^3 + b^3 = c^3$ . Prove that:  
 $a^2 + b^2 - c^2 > 6(c - a)(c - b)$ .

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