India National Olympiad 2010

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1 Let ABC be a triangle with circum-circle Γ. Let M be a point in the interior of triangle ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB; and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

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2 Find all natural numbers $n > 1$ such that n^2 does not divide $(n-2)!$.

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3 Find all non-zero real numbers x, y, z which satisfy the system of equations:

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) = xyz$$

$$(x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)(z^4+z^2x^2+x^4) = x^3y^3z^3$$

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4 How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set $\{1, 2, 3, 4\}$ and the six expressions

$$a_j^2 - a_ja_{j+1} + a_{j+1}^2$$

for $j = 1, 2, 3, 4, 5, 6$ (where a_7 is to be taken as a_1) are all equal to one another?

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5 Let ABC be an acute-angled triangle with altitude AK. Let H be its ortho-centre and O be its circum-centre. Suppose KOH is an acute-angled triangle and P its circum-centre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid-points of AB and AC.

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6 Define a sequence $< a_n >_{n\geq 0}$ by $a_0 = 0$, $a_1 = 1$ and

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$.
(a) For every $m > 0$ and $0 \leq j \leq m$, prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$.
(b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n.
