India National Olympiad 2011

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1 Let D, E, F be points on the sides BC, CA, AB respectively of a triangle ABC such that $BD = CE = AF$ and $\angle BDF = \angle CED = \angle AFE$. Show that $\triangle ABC$ is equilateral.

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2 Call a natural number n faithful if there exist natural numbers $a < b < c$ such that $a|b$, and $b|c$ and $n = a+b+c$. (i) Show that all but a finite number of natural numbers are faithful. (ii) Find the sum of all natural numbers which are not faithful.

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3 Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$ be two polynomials with integral coefficients such that $a_n - b_n$ is a prime and $a_n b_0 - a_0 b_n \neq 0$, and $a_{n-1} = b_{n-1}$. Suppose that there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that $r \in \mathbb{Z}$.

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4 Suppose five of the nine vertices of a regular nine-sided polygon are arbitrarily chosen. Show that one can select four among these five such that they are the vertices of a trapezium.

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5 Let $ABCD$ be a cyclic quadrilateral inscribed in a circle Γ. Let E, F, G, H be the midpoints of arcs AB, BC, CD, AD of Γ, respectively. Suppose that $AC \cdot BD = EG \cdot FH$. Show that AC, BD, EG, FH are all concurrent.

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6 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y)f(x-y) = (f(x)+f(y))^2 - 4x^2 f(y),$$

For all $x, y \in \mathbb{R}$.
