India National Olympiad 2012

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–	February 5th	

1 Let $ABCD$ be a quadrilateral inscribed in a circle. Suppose $AB = \sqrt{2+\sqrt{2}}$ and AB subtends 135 degrees at center of circle . Find the maximum possible area of $ABCD$.

2 Let $p_1 < p_2 < p_3 < p_4$ and $q_1 < q_2 < q_3 < q_4$ be two sets of prime numbers, such that $p_4 - p_1 = 8$ and $q_4 - q_1 = 8$. Suppose $p_1 > 5$ and $q_1 > 5$. Prove that 30 divides $p_1 - q_1$.

3 Define a sequence $< f_0(x), f_1(x), f_2(x), \cdots >$ of functions by

$$f_0(x) = 1$$

$$f_1(x) = x$$

$$(f_n(x))^2 - 1 = f_{n+1}(x) f_{n-1}(x)$$

for $n \geq 1$. Prove that each $f_n(x)$ is a polynomial with integer coefficients.

4 Let ABC be a triangle. An interior point P of ABC is said to be *good* if we can find exactly 27 rays emanating from P intersecting the sides of the triangle ABC such that the triangle is divided by these rays into 27 *smaller triangles of equal area.* Determine the number of good points for a given triangle ABC.

5 Let ABC be an acute angled triangle. Let D, E, F be points on BC, CA, AB such that AD is the median, BE is the internal bisector and CF is the altitude. Suppose that $\angle FDE = \angle C, \angle DEF = \angle A$ and $\angle EFD = \angle B$. Show that ABC is equilateral.

6 Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function satisfying $f(0) \neq 0$, $f(1) = 0$ and

$(i) f(xy) + f(x)f(y) = f(x) + f(y)$

$(ii) \left(f(x-y) - f(0)\right) f(x)f(y) = 0$

for all $x, y \in \mathbb{Z}$, simultaneously.

(a) Find the set of all possible values of the function f.

(b) If $f(10) \neq 0$ and $f(2) = 0$, find the set of all integers n such that $f(n) \neq 0$.