India National Olympiad 2014

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1 In a triangle ABC, let D be the point on the segment BC such that $AB+BD=AC+CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB=AC$.

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2 Let n be a natural number. Prove that,

$$\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor + \lfloor \sqrt{n} \rfloor$$

is even.

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3 Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisble by $a+b$. Prove that the quotient is at most $\frac{a+b}{4}$. When is this quotient exactly equal to $\frac{a+b}{4}$

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4 Written on a blackboard is the polynomial $x^2+x+2014$. Calvin and Hobbes take turns alternately (starting with Calvin) in the following game. At his turn, Calvin should either increase or decrease the coefficient of x by 1. And at this turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning stratergy.

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5 In a acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

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6 Let $n > 1$ be a natural number. Let $U = \{1, 2, ..., n\}$, and define $A\Delta B$ to be the set of all those elements of U which belong to exactly one of A and B. Show that $|\mathcal{F}| \leq 2^{n-1}$, where $\mathcal{F}$ is a collection of subsets of U such that for any two distinct elements of A, B of $\mathcal{F}$ we have $|A\Delta B| \geq 2$. Also find all such collections $\mathcal{F}$ for which the maximum is attained.
