Western Mathematical Olympiad 2017

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– Day 1

1 Let p be a prime and n be a positive integer such that p^2 divides $\prod_{k=1}^n(k^2+1)$. Show that $p<2n$.

2 Let n be a positive integer such that there exist positive integers $x_1, x_2, \cdots, x_n$ satisfying

$$x_1x_2\cdots x_n(x_1+x_2+\cdots+x_n)=100n.$$

Find the greatest possible value of n.

3 In triangle ABC, let D be a point on BC. Let I_1 and I_2 be the incenters of triangles ABD and ACD respectively. Let O_1 and O_2 be the circumcenters of triangles AI_1D and AI_2D respectively. Let lines I_1O_2 and I_2O_1 meet at P. Show that $PD \perp BC$.

4 Let n and k be given integers such that $n \geq k \geq 2$. Alice and Bob play a game on an n by n table with white cells. They take turns to pick a white cell and color it black. Alice moves first. The game ends as soon as there is at least one black cell in every k by k square after a player moves, who is declared the winner of the game. Who has the winning strategy?

– Day 2

5 Let $a_1, a_2, \cdots, a_9$ be 9 positive integers (not necessarily distinct) satisfying: for all $1 \leq i < j < k \leq 9$, there exists $l(1 \leq l \leq 9)$ distinct from i, j and j such that $a_i+a_j+a_k+a_l=100$. Find the number of 9-tuples $(a_1, a_2, \cdots, a_9)$ satisfying the above conditions.

6 In acute triangle ABC, let D and E be points on sides AB and AC respectively. Let segments BE and DC meet at point H. Let M and N be the midpoints of segments BD and CE respectively. Show that H is the orthocenter of triangle AMN if and only if B, C, E, D are concyclic and $BE \perp CD$.

7 Let $n = 2^\alpha \cdot q$ be a positive integer, where α is a nonnegative integer and q is an odd number. Show that for any positive integer m, the number of integer solutions to the equation $x_1^2+x_2^2+\cdots+x_n^2=m$ is divisible by $2^{\alpha+1}$.

8 Let $a_1, a_2, \cdots, a_n > 0$ $(n \geq 2)$. Prove that

$$\sum_{i=1}^n max\{a_1, a_2, \cdots, a_i\} \cdot min\{a_i, a_{i+1}, \cdots, a_n\} \leq \frac{n}{2\sqrt{n-1}} \sum_{i=1}^n a_i^2$$

