China Team Selection Test 1986

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Day 1

1 If $ABCD$ is a cyclic quadrilateral, then prove that the incenters of the triangles ABC, BCD, CDA, DAB are the vertices of a rectangle.

2 Let a_1, a_2, ..., a_n and b_1, b_2, ..., b_n be $2 \cdot n$ real numbers. Prove that the following two statements are equivalent:

i) For any n real numbers x_1, x_2, ..., x_n satisfying $x_1 \leq x_2 \leq \ldots \leq x_n$, we have $\sum^{n}_{k=1} a_k \cdot x_k \leq \sum^{n}_{k=1} b_k \cdot x_k,$

ii) We have $\sum^{s}_{k=1} a_k \leq \sum^{s}_{k=1} b_k$ for every $s \in \{1, 2, ..., n-1\}$ and $\sum^{n}_{k=1} a_k = \sum^{n}_{k=1} b_k$.

3 Given a positive integer A written in decimal expansion: $(a_n, a_{n-1}, \ldots, a_0)$ and let $f(A)$ denote $\sum^{n}_{k=0} 2^{n-k} \cdot a_k$. Define $A_1 = f(A), A_2 = f(A_1)$. Prove that:

I. There exists positive integer k for which $A_{k+1} = A_k$.
II. Find such A_k for 19^{86}.

4 Given a triangle ABC for which $C = 90$ degrees, prove that given n points inside it, we can name them $P_1, P_2, \ldots , P_n$ in some way such that:
$\sum^{n-1}_{k=1} \left(P_K P_{k+1} \right)^2 \leq AB^2$ (the sum is over the consecutive square of the segments from 1 up to $n-1$).

Edited by orl.

Day 2

1 Given a square $ABCD$ whose side length is 1, P and Q are points on the sides AB and AD. If the perimeter of APQ is 2 find the angle PCQ.

2 Given a tetrahedron $ABCD$, E, F, G, are on the respectively on the segments AB, AC and AD. Prove that:

i) area $EFG \leq$ maxarea ABC,area ABD,area ACD,area BCD.
ii) The same as above replacing "area" for "perimeter".

3 Let $x_i, 1 \leq i \leq n$ be real numbers with $n \geq 3$. Let p and q be their symmetric sum of degree 1 and 2 respectively. Prove that:

i) $p^2 \cdot \frac{n-1}{n} - 2q \geq 0$

ii) $\left|x_i - \frac{p}{n}\right| \leq \sqrt{p^2 - \frac{2nq}{n-1}} \cdot \frac{n-1}{n}$ for every meaningful i.

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4 Mark $4 \cdot k$ points in a circle and number them arbitrarily with numbers from 1 to $4 \cdot k$. The chords cannot share common endpoints, also, the endpoints of these chords should be among the $4 \cdot k$ points.

i. Prove that $2 \cdot k$ pairwisely non-intersecting chords can be drawn for each of whom its end-points differ in at most $3 \cdot k - 1$.
ii. Prove that the $3 \cdot k - 1$ cannot be improved.
