China Team Selection Test 1987

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Day 1

1 a.) For all positive integer k find the smallest positive integer $f(k)$ such that 5 sets $s_1, s_2, \ldots, s_5$ exist satisfying:

i. each has k elements;
ii. s_i and s_{i+1} are disjoint for $i = 1, 2, ..., 5$ ($s_6 = s_1$)
iii. the union of the 5 sets has exactly $f(k)$ elements.

b.) Generalisation: Consider $n \geq 3$ sets instead of 5.

2 A closed recticular polygon with 100 sides (may be concave) is given such that it's vertices have integer coordinates, it's sides are parallel to the axis and all it's sides have odd length. Prove that it's area is odd.

3 Let $r_1 = 2$ and $r_n = \prod_{k=1}^{n-1} r_i + 1$, $n \geq 2$. Prove that among all sets of positive integers such that $\sum_{k=1}^{n} \frac{1}{a_i} < 1$, the partial sequences $r_1, r_2, ..., r_n$ are the one that gets nearer to 1.

Day 2

1 Given a convex figure in the Cartesian plane that is symmetric with respect of both axis, we construct a rectangle A inside it with maximum area (over all posible rectangles). Then we enlarge it with center in the center of the rectangle and ratio lamda such that is covers the convex figure. Find the smallest lamda such that it works for all convex figures.

2 Find all positive integer n such that the equation $x^3 + y^3 + z^3 = n \cdot x^2 \cdot y^2 \cdot z^2$ has positive integer solutions.

3 Let G be a simple graph with $2 \cdot n$ vertices and $n^2 + 1$ edges. Show that this graph G contains a $K_4 -$ one edge, that is, two triangles with a common edge.