China Team Selection Test 1988

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Day 1

1 Suppose real numbers A, B, C such that for all real numbers x, y, z the following inequality holds:

$$A(x-y)(x-z) + B(y-z)(y-x) + C(z-x)(z-y) \geq 0.$$

Find the necessary and sufficient condition A, B, C must satisfy (expressed by means of an equality or an inequality).

2 Find all functions $f : \mathbb{Q} \mapsto \mathbb{C}$ satisfying

(i) For any $x_1, x_2, \ldots, x_{1988} \in \mathbb{Q}$, $f(x_1 + x_2 + \ldots + x_{1988}) = f(x_1)f(x_2)\ldots f(x_{1988})$.

(ii) $\overline{f(1988)}f(x) = f(1988)\overline{f(x)}$ for all $x \in \mathbb{Q}$.

3 In triangle ABC, $\angle C = 30^\circ$, O and I are the circumcenter and incenter respectively, Points $D \in AC$ and $E \in BC$, such that $AD = BE = AB$. Prove that $OI = DE$ and $OI \perp DE$.

4 Let $k \in \mathbb{N}$, $S_k = \{(a, b) | a, b = 1, 2, \ldots, k\}$. Any two elements (a, b), $(c, d) \in S_k$ are called "undistinguishing" in S_k if $a - c \equiv 0$ or $\pm 1 \pmod{k}$ and $b - d \equiv 0$ or $\pm 1 \pmod{k}$; otherwise, we call them "distinguishing". For example, $(1, 1)$ and $(2, 5)$ are undistinguishing in S_5. Considering the subset A of S_k such that the elements of A are pairwise distinguishing. Let r_k be the maximum possible number of elements of A.

(i) Find r_5.
(ii) Find r_7.
(iii) Find r_k for $k \in \mathbb{N}$.

Day 2

1 Let $f(x) = 3x + 2$. Prove that there exists $m \in \mathbb{N}$ such that $f^{100}(m)$ is divisible by 1988.

2 Let $ABCD$ be a trapezium $AB//CD$, M and N are fixed points on AB, P is a variable point on CD. $E = DN \cap AP$, $F = DN \cap MC$, $G = MC \cap PB$, $DP = \lambda \cdot CD$. Find the value of λ for which the area of quadrilateral $PEFG$ is maximum.

3 A polygon $\prod$ is given in the OXY plane and its area exceeds n. Prove that there exist $n+1$ points $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_{n+1}(x_{n+1}, y_{n+1})$ in $\prod$ such that $\forall i, j \in \{1, 2, \ldots, n+1\}$, $x_j - x_i$ and $y_j - y_i$ are all integers.

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4 There is a broken computer such that only three primitive data c, 1 and -1 are reserved. Only allowed operation may take u and v and output $u \cdot v + v$. At the beginning, $u, v \in \{c, 1, -1\}$. After then, it can also take the value of the previous step (only one step back) besides $\{c, 1, -1\}$. Prove that for any polynomial $P_n(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + \ldots + a_n$ with integer coefficients, the value of $P_n(c)$ can be computed using this computer after only finite operation.
