China Team Selection Test 1989

www.artofproblemsolving.com/community/c4945

by orl, Rafal, harazi, yetti, grobber, darij grinberg, seshadri

Day 1

1 A triangle of sides $\frac{3}{2}, \frac{\sqrt{5}}{2}, \sqrt{2}$ is folded along a variable line perpendicular to the side of $\frac{3}{2}$. Find the maximum value of the coincident area.

2 Let $v_0 = 0, v_1 = 1$ and $v_{n+1} = 8 \cdot v_n - v_{n-1}, n = 1, 2, ...$ Prove that in the sequence $\{v_n\}$ there aren't terms of the form $3^\alpha \cdot 5^\beta$ with $\alpha, \beta \in \mathbb{N}$.

3 Find the greatest n such that $(z+1)^n = z^n + 1$ has all its non-zero roots in the unitary circumference, e.g. $(\alpha+1)^n = \alpha^n + 1, \alpha \neq 0$ implies $|\alpha| = 1$.

4 Given triangle ABC, squares $ABEF, BCGH, CAIJ$ are constructed externally on side AB, BC, CA, respectively. Let $AH \cap BJ = P_1$, $BJ \cap CF = Q_1$, $CF \cap AH = R_1$, $AG \cap CE = P_2$, $BI \cap AG = Q_2$, $CE \cap BI = R_2$. Prove that triangle $P_1Q_1R_1$ is congruent to triangle $P_2Q_2R_2$.

Day 2

1 Let $\mathbb{N} = \{1, 2, \ldots\}$. Does there exists a function $f : \mathbb{N} \mapsto \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $f^{1989}(n) = 2 \cdot n$?

2 AD is the altitude on side BC of triangle ABC. If $BC + AD - AB - AC = 0$, find the range of $\angle BAC$.

Alternative formulation. Let AD be the altitude of triangle ABC to the side BC. If $BC + AD = AB + AC$, then find the range of $\angle A$.

3 1989 equal circles are arbitrarily placed on the table without overlap. What is the least number of colors are needed such that all the circles can be painted with any two tangential circles colored differently.

4 $\forall n \in \mathbb{N}$, $P(n)$ denotes the number of the partition of n as the sum of positive integers (disregarding the order of the parts), e.g. since $4 = 1+1+1+1 = 1+1+2 = 1+3 = 2+2 = 4$, so $P(4) = 5$. "Dispersion" of a partition denotes the number of different parts in that partitation. And denote $q(n)$ is the sum of all the dispersions, e.g. $q(4) = 1+2+2+1+1 = 7$. $n \geq 1$. Prove that

(1) $q(n) = 1 + \sum_{i=1}^{n-1} P(i)$.

(2) $1 + \sum_{i=1}^{n-1} P(i) \leq \sqrt{2} \cdot n \cdot P(n)$.