China Team Selection Test 1990

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Day 1

1 In a wagon, every $m \geq 3$ people have exactly one common friend. (When A is B's friend, B is also A's friend. No one was considered as his own friend.) Find the number of friends of the person who has the most friends.

2 Finitely many polygons are placed in the plane. If for any two polygons of them, there exists a line through origin O that cuts them both, then these polygons are called "properly placed". Find the least $m \in \mathbb{N}$, such that for any group of properly placed polygons, m lines can drawn through O and every polygon is cut by at least one of these m lines.

3 In set S, there is an operation $''\circ''$ such that $\forall a, b \in S$, a unique $a \circ b \in S$ exists. And

(i) $\forall a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$.

(ii) $a \circ b \neq b \circ a$ when $a \neq b$.

Prove that:

a.) $\forall a, b, c \in S$, $(a \circ b) \circ c = a \circ c$.

b.) If $S = \{1, 2, \ldots, 1990\}$, try to define an operation $''\circ''$ in S with the above properties.

4 Number a is such that $\forall a_1, a_2, a_3, a_4 \in \mathbb{R}$, there are integers k_1, k_2, k_3, k_4 such that $\sum_{1 \leq i < j \leq 4} ((a_i - k_i) - (a_j - k_j))^2 \leq a$. Find the minimum of a.

Day 2

1 Given a triangle ABC with angle $C \geq 60^\circ$. Prove that:
$(a+b) \cdot \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 4 + \frac{1}{\sin\left(\frac{C}{2}\right)}$.

2 Find all functions $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ such that $f(x) - g(y) = (x-y) \cdot h(x+y)$ for $x, y \in \mathbb{R}$.

3 Prove that for every integer power of 2, there exists a multiple of it with all digits (in decimal expression) not zero.

4 There are arbitrary 7 points in the plane. Circles are drawn through every 4 possible concyclic points. Find the maximum number of circles that can be drawn.