China Team Selection Test 1991

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Day 1

1 Let real coefficient polynomial $f(x) = x^n + a_1 \cdot x^{n-1} + \ldots + a_n$ has real roots $b_1, b_2, \ldots, b_n$, $n \geq 2$, prove that $\forall x \geq max\{b_1, b_2, \ldots, b_n\}$, we have

$$f(x+1) \geq \frac{2 \cdot n^2}{\frac{1}{x-b_1} + \frac{1}{x-b_2} + \ldots + \frac{1}{x-b_n}}.$$

2 For $i = 1, 2, \ldots, 1991$, we choose n_i points and write number i on them (each point has only written one number on it). A set of chords are drawn such that:

(i) They are pairwise non-intersecting.
(ii) The endpoints of each chord have distinct numbers.

If for all possible assignments of numbers the operation can always be done, find the necessary and sufficient condition the numbers $n_1, n_2, \ldots, n_{1991}$ must satisfy for this to be possible.

3 5 points are given in the plane, any three non-collinear and any four non-concyclic. If three points determine a circle that has one of the remaining points inside it and the other one outside it, then the circle is said to be *good*. Let the number of good circles be n; find all possible values of n.

Day 2

1 We choose 5 points $A_1, A_2, \ldots, A_5$ on a circumference of radius 1 and centre O. P is a point inside the circle. Denote Q_i as the intersection of A_iA_{i+2} and $A_{i+1}P$, where $A_7 = A_2$ and $A_6 = A_1$. Let $OQ_i = d_i, i = 1, 2, \ldots, 5$. Find the product $\prod_{i=1}^5 A_iQ_i$ in terms of d_i.

2 Let f be a function $f : \mathbb{N} \cup \{0\} \mapsto \mathbb{N}$, and satisfies the following conditions:

(1) $f(0) = 0, f(1) = 1,$
(2) $f(n+2) = 23 \cdot f(n+1) + f(n), n = 0, 1, \ldots.$

Prove that for any $m \in \mathbb{N}$, there exist a $d \in \mathbb{N}$ such that $m | f(f(n)) \Leftrightarrow d | n.$

3 All edges of a polyhedron are painted with red or yellow. For an angle of a facet, if the edges determining it are of different colors, then the angle is called *excentric*. The *excentricity* of a vertex A, namely S_A, is defined as the number of excentric angles it has. Prove that there exist two vertices B and C such that $S_B + S_C \leq 4$.