China Team Selection Test 1992

www.artofproblemsolving.com/community/c4948

by orl, Pascual2005, grobber, yetti

Day 1

1 16 students took part in a competition. All problems were multiple choice style. Each problem had four choices. It was said that any two students had at most one answer in common, find the maximum number of problems.

2 Let $n \geq 2, n \in \mathbb{N}$, find the least positive real number λ such that for arbitrary $a_i \in \mathbb{R}$ with $i = 1, 2, \ldots, n$ and $b_i \in \left[0, \frac{1}{2}\right]$ with $i = 1, 2, \ldots, n$, the following holds:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1 \Rightarrow \prod_{i=1}^{n} a_i \leq \lambda \sum_{i=1}^{n} a_i b_i.$$

3 For any prime p, prove that there exists integer x_0 such that $p|(x_0^2 - x_0 + 3) \Leftrightarrow$ there exists integer y_0 such that $p|(y_0^2 - y_0 + 25)$.

Day 2

1 A triangle ABC is given in the plane with $AB = \sqrt{7}$, $BC = \sqrt{13}$ and $CA = \sqrt{19}$, circles are drawn with centers at A, B and C and radii $\frac{1}{3}$, $\frac{2}{3}$ and 1, respectively. Prove that there are points A', B', C' on these three circles respectively such that triangle ABC is congruent to triangle $A'B'C'$.

2 A $(3n + 1) \times (3n + 1)$ table $(n \in \mathbb{N})$ is given. Prove that deleting any one of its squares yields a shape cuttable into pieces of the following form and its rotations: "L" shape formed by cutting one square from a 2×2 squares.

3 For any $n, T \geq 2, n, T \in \mathbb{N}$, find all $a \in \mathbb{N}$ such that $\forall a_i > 0, i = 1, 2, \ldots, n$, we have

$$\sum_{k=1}^{n} \frac{a \cdot k + \frac{a^2}{4}}{S_k} < T^2 \cdot \sum_{k=1}^{n} \frac{1}{a_k},$$

where $S_k = \sum_{i=1}^{k} a_i$.