China Team Selection Test 1993

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Day 1

1 For all primes $p \geq 3$, define $F(p) = \sum_{k=1}^{\frac{p-1}{2}} k^{120}$ and $f(p) = \frac{1}{2} - \left\{ \frac{F(p)}{p} \right\}$, where $\{x\} = x - [x]$, find the value of $f(p)$.

2 Let $n \geq 2, n \in \mathbb{N}$, $a, b, c, d \in \mathbb{N}$, $\frac{a}{b} + \frac{c}{d} < 1$ and $a + c \leq n$, find the maximum value of $\frac{a}{b} + \frac{c}{d}$ for fixed n.

3 A graph $G = (V, E)$ is given. If at least n colors are required to paints its vertices so that between any two same colored vertices no edge is connected, then call this graph "$n-$colored". Prove that for any $n \in \mathbb{N}$, there is a $n-$colored graph without triangles.

Day 2

1 Find all integer solutions to $2x^4 + 1 = y^2$.

2 Let $S = \{(x, y) | x = 1, 2, \ldots, 1993, y = 1, 2, 3, 4\}$. If $T \subset S$ and there aren't any squares in T. Find the maximum possible value of $|T|$. The squares in T use points in S as vertices.

3 Let ABC be a triangle and its bisector at A cuts its circumcircle at D. Let I be the incenter of triangle ABC, M be the midpoint of BC, P is the symmetric to I with respect to M (Assuming P is in the circumcircle). Extend DP until it cuts the circumcircle again at N. Prove that among segments AN, BN, CN, there is a segment that is the sum of the other two.