China Team Selection Test 1994

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by orl, Pascual2005, fdump, Fedor Petrov, darij grinberg

Day 1

1 Find all sets comprising of 4 natural numbers such that the product of any 3 numbers in the set leaves a remainder of 1 when divided by the remaining number.

2 An n by n grid, where every square contains a number, is called an n-code if the numbers in every row and column form an arithmetic progression. If it is sufficient to know the numbers in certain squares of an n-code to obtain the numbers in the entire grid, call these squares a key.

a.) Find the smallest $s \in \mathbb{N}$ such that any s squares in an $n-$code $(n \geq 4)$ form a key.

b.) Find the smallest $t \in \mathbb{N}$ such that any t squares along the diagonals of an n-code $(n \geq 4)$ form a key.

3 Find the smallest $n \in \mathbb{N}$ such that if any 5 vertices of a regular n-gon are colored red, there exists a line of symmetry l of the n-gon such that every red point is reflected across l to a non-red point.

Day 2

1 Given $5n$ real numbers $r_i, s_i, t_i, u_i, v_i \geq 1 (1 \leq i \leq n)$, let $R = \frac{1}{n}\sum_{i=1}^{n} r_i$, $S = \frac{1}{n}\sum_{i=1}^{n} s_i$, $T = \frac{1}{n}\sum_{i=1}^{n} t_i$, $U = \frac{1}{n}\sum_{i=1}^{n} u_i$, $V = \frac{1}{n}\sum_{i=1}^{n} v_i$. Prove that $\prod_{i=1}^{n}\frac{r_i s_i t_i u_i v_i + 1}{r_i s_i t_i u_i v_i - 1} \geq \left(\frac{RSTUV+1}{RSTUV-1}\right)^n$.

2 Given distinct prime numbers p and q and a natural number $n \geq 3$, find all $a \in \mathbb{Z}$ such that the polynomial $f(x) = x^n + ax^{n-1} + pq$ can be factored into 2 integral polynomials of degree at least 1.

3 For any 2 convex polygons S and T, if all the vertices of S are vertices of T, call S a sub-polygon of T.

I. Prove that for an odd number $n \geq 5$, there exists m sub-polygons of a convex n-gon such that they do not share any edges, and every edge and diagonal of the n-gon are edges of the m sub-polygons.

II. Find the smallest possible value of m.