China Team Selection Test 1995

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by orl, Pascual2005, grobber, kevinatcausa, fleeting_guest

Day 1

1 Find the smallest prime number p that cannot be represented in the form $|3^{a} - 2^{b}|$, where a and b are non-negative integers.

2 Given a fixed acute angle θ and a pair of internally tangent circles, let the line l which passes through the point of tangency, A, cut the larger circle again at B (l does not pass through the centers of the circles). Let M be a point on the major arc AB of the larger circle, N the point where AM intersects the smaller circle, and P the point on ray MB such that $\angle MPN = \theta$. Find the locus of P as M moves on major arc AB of the larger circle.

3 21 people take a test with 15 true or false questions. It is known that every 2 people have at least 1 correct answer in common. What is the minimum number of people that could have correctly answered the question which the most people were correct on?

Day 2

1 Let $S = \{A = (a_1, \ldots, a_s) \mid a_i = 0$ or $1, i = 1, \ldots, 8\}$. For any 2 elements of S, $A = \{a_1, \ldots, a_8\}$ and $B = \{b_1, \ldots, b_8\}$. Let $d(A, B) = \sum_{i=1} 8|a_i - b_i|$. Call $d(A, B)$ the distance between A and B. At most how many elements can S have such that the distance between any 2 sets is at least 5?

2 A and B play the following game with a polynomial of degree at least 4:

$$x^{2n} + \_x^{2n-1} + \_x^{2n-2} + \ldots + \_x + 1 = 0$$

A and B take turns to fill in one of the blanks with a real number until all the blanks are filled up. If the resulting polynomial has no real roots, A wins. Otherwise, B wins. If A begins, which player has a winning strategy?

3 Prove that the interval $[0, 1]$ can be split into black and white intervals for any quadratic polynomial $P(x)$, such that the sum of weights of the black intervals is equal to the sum of weights of the white intervals. (Define the weight of the interval $[a, b]$ as $P(b) - P(a)$.)

Does the same result hold with a degree 3 or degree 5 polynomial?