China Team Selection Test 1996

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Day 1

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1 Let side BC of $\triangle ABC$ be the diameter of a semicircle which cuts AB and AC at D and E respectively. F and G are the feet of the perpendiculars from D and E to BC respectively. DG and EF intersect at M. Prove that $AM \perp BC$.

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2 S is the set of functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy the following conditions:

I. $f(1) = 2$
II. $f(n+1) \geq f(n) \geq \frac{n}{n+1} f(2n)$ for $n = 1, 2, \ldots$

Find the smallest $M \in \mathbb{N}$ such that for any $f \in S$ and any $n \in \mathbb{N}$, $f(n) < M$.

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3 Let $M = \{2, 3, 4, \ldots 1000\}$. Find the smallest $n \in \mathbb{N}$ such that any n-element subset of M contains 3 pairwise disjoint 4-element subsets S, T, U such that

I. For any 2 elements in S, the larger number is a multiple of the smaller number. The same applies for T and U.

II. For any $s \in S$ and $t \in T$, $(s, t) = 1$.

III. For any $s \in S$ and $u \in U$, $(s, u) > 1$.

Day 2

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1 3 countries A, B, C participate in a competition where each country has 9 representatives. The rules are as follows: every round of competition is between 1 competitor each from 2 countries. The winner plays in the next round, while the loser is knocked out. The remaining country will then send a representative to take on the winner of the previous round. The competition begins with A and B sending a competitor each. If all competitors from one country have been knocked out, the competition continues between the remaining 2 countries until another country is knocked out. The remaining team is the champion.

I. At least how many games does the champion team win?

II. If the champion team won 11 matches, at least how many matches were played?

2 Let $\alpha_1, \alpha_2, \dots, \alpha_n$, and $\beta_1, \beta_2, \dots, \beta_n$, where $n \geq 4$, be 2 sets of real numbers such that

$$\sum_{i=1}^{n} \alpha_i^2 < 1 \quad \text{and} \quad \sum_{i=1}^{n} \beta_i^2 < 1.$$

Define

$$A^2 = 1 - \sum_{i=1}^{n} \alpha_i^2,$$
$$B^2 = 1 - \sum_{i=1}^{n} \beta_i^2,$$
$$W = \frac{1}{2}(1 - \sum_{i=1}^{n} \alpha_i \beta_i)^2.$$

Find all real numbers λ such that the polynomial

$$x^n + \lambda(x^{n-1} + \cdots + x^3 + Wx^2 + ABx + 1) = 0,$$

only has real roots.

3 Does there exist non-zero complex numbers a, b, c and natural number h such that if integers k, l, m satisfy $|k| + |l| + |m| \geq 1996$, then $|ka + lb + mc| > \frac{1}{h}$ is true?