China Team Selection Test 1997

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Day 1

1 Given a real number $\lambda > 1$, let P be a point on the arc BAC of the circumcircle of $\triangle ABC$. Extend BP and CP to U and V respectively such that $BU = \lambda BA$, $CV = \lambda CA$. Then extend UV to Q such that $UQ = \lambda UV$. Find the locus of point Q.

2 There are n football teams in a round-robin competition where every 2 teams meet once. The winner of each match receives 3 points while the loser receives 0 points. In the case of a draw, both teams receive 1 point each. Let k be as follows: $2 \leq k \leq n - 1$. At least how many points must a certain team get in the competition so as to ensure that there are at most $k - 1$ teams whose scores are not less than that particular team's score?

3 Prove that there exists $m \in \mathbb{N}$ such that there exists an integral sequence $\{a_n\}$ which satisfies:

I. $a_0 = 1, a_1 = 337$;

II. $(a_{n+1}a_{n-1} - a_n^2) + \frac{3}{4}(a_{n+1} + a_{n-1} - 2a_n) = m, \forall\, n \geq 1$;

III. $\frac{1}{6}(a_n + 1)(2a_n + 1)$ is a perfect square $\forall\, n \geq 1$.

Day 2

1 Find all real-coefficient polynomials $f(x)$ which satisfy the following conditions:

i. $f(x) = a_0x^{2n} + a_2x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n}, a_0 > 0$;

ii. $\sum_{j=0}^{n} a_{2j}a_{2n-2j} \leq \left(\begin{array}{c} 2n \\ n \end{array} \right) a_0a_{2n}$;

iii. All the roots of $f(x)$ are imaginary numbers with no real part.

2 Let n be a natural number greater than 6. X is a set such that $|X| = n$. $A_1, A_2, \ldots, A_m$ are distinct 5-element subsets of X. If $m > \frac{n(n-1)(n-2)(n-3)(4n-15)}{600}$, prove that there exists $A_{i_1}, A_{i_2}, \ldots, A_{i_6}$ $(1 \leq i_1 < i_2 < \cdots, i_6 \leq m)$, such that $\bigcup_{k=1}^{6} A_{i_k} = 6$.

3 There are 1997 pieces of medicine. Three bottles A, B, C can contain at most 1997, 97, 19 pieces of medicine respectively. At first, all 1997 pieces are placed in bottle A, and the three bottles are closed. Each piece of medicine can be split into 100 part. When a bottle is opened, all pieces of medicine in that bottle lose a part each. A man wishes to consume all the medicine. However, he can only open each of the bottles at most once each day, consume one piece of medicine, move some pieces between the bottles, and close them. At least how many parts will be lost by the time he finishes consuming all the medicine?
