China Team Selection Test 1998

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Day 1

1 Find $k \in \mathbb{N}$ such that

a.) For any $n \in \mathbb{N}$, there does not exist $j \in \mathbb{Z}$ which satisfies the conditions $0 \leq j \leq n - k + 1$ and $\binom{n}{j}, \binom{n}{j+1}, \ldots, \binom{n}{j+k-1}$ forms an arithmetic progression.

b.) There exists $n \in \mathbb{N}$ such that there exists j which satisfies $0 \leq j \leq n - k + 2$, and $\binom{n}{j}, \binom{n}{j+1}, \ldots, \binom{n}{j+k-2}$ forms an arithmetic progression.

Find all n which satisfies part **b.)**

2 $n \geq 5$ football teams participate in a round-robin tournament. For every game played, the winner receives 3 points, the loser receives 0 points, and in the event of a draw, both teams receive 1 point. The third-from-bottom team has fewer points than all the teams ranked before it, and more points than the last 2 teams; it won more games than all the teams before it, but fewer games than the 2 teams behind it. Find the smallest possible n.

3 For a fixed $\theta \in [0, \frac{\pi}{2}]$, find the smallest $a \in \mathbb{R}^+$ which satisfies the following conditions:

I. $\frac{\sqrt{a}}{\cos\theta} + \frac{\sqrt{a}}{\sin\theta} > 1$.

II. There exists $x \in [1 - \frac{\sqrt{a}}{\sin\theta}, \frac{\sqrt{a}}{\cos\theta}]$ such that $[(1-x)\sin\theta - \sqrt{a - x^2\cos^2\theta}]^2 + [x\cos\theta - \sqrt{a - (1-x)^2\sin^2\theta}]^2 \leq a$.

Day 2

1 In acute-angled $\triangle ABC$, H is the orthocenter, O is the circumcenter and I is the incenter. Given that $\angle C > \angle B > \angle A$, prove that I lies within $\triangle BOH$.

2 Let n be a natural number greater than 2. l is a line on a plane. There are n distinct points P_1, P_2, , P_n on l. Let the product of distances between P_i and the other $n-1$ points be d_i ($i = 1, 2,$, n). There exists a point Q, which does not lie on l, on the plane. Let the distance from Q to P_i be C_i ($i = 1, 2, , n$). Find $S_n = \sum_{i=1}^{n} (-1)^{n-i} \frac{c_i^2}{d_i}$.

3 For any $h = 2^r$ (r is a non-negative integer), find all $k \in \mathbb{N}$ which satisfy the following condition: There exists an odd natural number $m > 1$ and $n \in \mathbb{N}$, such that $k \mid m^h - 1, m \mid n^{\frac{m^h-1}{k}} + 1$.