China Team Selection Test 1999

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Day 1

1 For non-negative real numbers $x_1, x_2, \ldots, x_n$ which satisfy $x_1 + x_2 + \cdots + x_n = 1$, find the largest possible value of $\sum_{j=1}^n (x_j^4 - x_j^5)$.

2 Find all prime numbers p which satisfy the following condition: For any prime $q < p$, if $p = kq + r, 0 \leq r < q$, there does not exist an integer $q > 1$ such that $a^2 \mid r$.

3 Let $S = \{1, 2, \ldots, 15\}$. Let $A_1, A_2, \ldots, A_n$ be n subsets of S which satisfy the following conditions:

I. $|A_i| = 7, i = 1, 2, \ldots, n$;

II. $|A_i \cap A_j| \leq 3, 1 \leq i < j \leq n$

III. For any 3-element subset M of S, there exists A_k such that $M \subset A_k$.

Find the smallest possible value of n.

Day 2

1 A circle is tangential to sides AB and AD of convex quadrilateral $ABCD$ at G and H respectively, and cuts diagonal AC at E and F. What are the necessary and sufficient conditions such that there exists another circle which passes through E and F, and is tangential to DA and DC extended?

2 For a fixed natural number $m \geq 2$, prove that

a.) There exists integers $x_1, x_2, \ldots, x_{2m}$ such that

$$x_i x_{m+i} = x_{i+1} x_{m+i-1} + 1, i = 1, 2, \ldots, m \qquad (*)$$

b.) For any set of integers $\{x_1, x_2, \ldots, x_{2m}$ which fulfils (*), an integral sequence $\ldots, y_{-k}, \ldots, y_{-1}, y_0, y_1, \ldots,$

can be constructed such that $y_k y_{m+k} = y_{k+1} y_{m+k-1} + 1, k = 0, \pm 1, \pm 2, \dots$ such that $y_i = x_i, i = 1, 2, \dots, 2m$.

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3 For every permutation τ of $1, 2, \dots, 10$, $\tau = (x_1, x_2, \dots, x_{10})$, define $S(\tau) = \sum_{k=1}^{10} |2x_k - 3x_{k-1}|$. Let $x_{11} = x_1$. Find

I. The maximum and minimum values of $S(\tau)$.

II. The number of τ which lets $S(\tau)$ attain its maximum.

III. The number of τ which lets $S(\tau)$ attain its minimum.
