China Team Selection Test 2001

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Day 1

1 E and F are interior points of convex quadrilateral $ABCD$ such that $AE = BE$, $CE = DE$, $\angle AEB = \angle CED$, $AF = DF$, $BF = CF$, $\angle AFD = \angle BFC$. Prove that $\angle AFD + \angle AEB = \pi$.

2 a and b are natural numbers such that $b > a > 1$, and a does not divide b. The sequence of natural numbers $\{b_n\}_{n=1}^\infty$ satisfies $b_{n+1} \geq 2b_n \forall n \in \mathbb{N}$. Does there exist a sequence $\{a_n\}_{n=1}^\infty$ of natural numbers such that for all $n \in \mathbb{N}$, $a_{n+1} - a_n \in \{a, b\}$, and for all $m, l \in \mathbb{N}$ (m may be equal to l), $a_m + a_l \not\in \{b_n\}_{n=1}^\infty$?

3 For a given natural number $k > 1$, find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f[x^k + f(y)] = y + [f(x)]^k$.

Day 2

1 For a given natural number $n > 3$, the real numbers $x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}$ satisfy the conditions $0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < x_{n+2}$. Find the minimum possible value of

$$\frac{(\sum_{i=1}^n \frac{x_{i+1}}{x_i})(\sum_{j=1}^n \frac{x_{j+2}}{x_{j+1}})}{(\sum_{k=1}^n \frac{x_{k+1}x_{k+2}}{x_{k+1}^2+x_kx_{k+2}})(\sum_{l=1}^n \frac{x_{l+1}^2+x_lx_{l+2}}{x_lx_{l+1}})}$$

and find all $(n+2)$-tuplets of real numbers $(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2})$ which gives this value.

2 In the equilateral $\triangle ABC$, D is a point on side BC. O_1 and I_1 are the circumcenter and incenter of $\triangle ABD$ respectively, and O_2 and I_2 are the circumcenter and incenter of $\triangle ADC$ respectively. O_1I_1 intersects O_2I_2 at P. Find the locus of point P as D moves along BC.

3 Let $F = \max_{1 \leq x \leq 3} |x^3 - ax^2 - bx - c|$. When a, b, c run over all the real numbers, find the smallest possible value of F.