China Team Selection Test 2002

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Day 1

1 Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let O be the foot of the perpendicular from P to EF. Show that $\angle BOC = \angle AOD$.

2 Let (a_n) be the sequence of reals defined by $a_1 = \frac{1}{4}$ and the recurrence $a_n = \frac{1}{4}(1+a_{n-1})^2, n \geq 2$. Find the minimum real λ such that for any non-negative reals $x_1, x_2, \ldots, x_{2002}$, it holds

$$\sum_{k=1}^{2002} A_k \leq \lambda a_{2002},$$

where $A_k = \frac{x_k - k}{(x_k + \cdots + x_{2002} + \frac{k(k-1)}{2} + 1)^2}, k \geq 1$.

3 Seventeen football fans were planning to go to Korea to watch the World Cup football match. They selected 17 matches. The conditions of the admission tickets they booked were such that

- One person should book at most one admission ticket for one match;

- At most one match was same in the tickets booked by every two persons;

- There was one person who booked six tickets.

How many tickets did those football fans book at most?

Day 2

1 Find all natural numbers $n(n \geq 2)$ such that there exists reals $a_1, a_2, \ldots, a_n$ which satisfy

$$\{|a_i - a_j| \mid 1 \leq i < j \leq n\} = \left\{1, 2, \ldots, \frac{n(n-1)}{2}\right\}.$$

Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{7, 8, 9, \ldots, n\}$. $A_i(i = 1, 2, \ldots, 20)$ contains eight numbers, three of which are chosen from A and the other five numbers from B. $|A_i \cap A_j| \leq 2, 1 \leq i < j \leq 20$. Find the minimum possible value of n.

2   Given an integer k. $f(n)$ is defined on negative integer set and its values are integers. $f(n)$ satisfies

$$f(n)f(n+1) = (f(n)+n-k)^2,$$

for $n = -2, -3, \cdots$. Find an expression of $f(n)$.

3   Let

$$f(x_1, x_2, x_3) = -2 \cdot (x_1^3 + x_2^3 + x_3^3) + 3 \cdot (x_1^2(x_2 + x_3) + x_2^2 \cdot (x_1 + x_3) + x_3^2 \cdot (x_1 + x_2) - 12x_1x_2x_3.$$

For any reals r, s, t, we denote

$$g(r, s, t) = \max_{t \leq x_3 \leq t+2} |f(r, r+2, x_3) + s|.$$

Find the minimum value of $g(r, s, t)$.

–   Quiz 1

1   Given $n \geq 3$, n is a integer. Prove that:

$$(2^n - 2) \cdot \sqrt{2i-1} \geq \left(\sum_{j=0}^{i-1} C_n^j + C_{n-1}^{i-1} \right) \cdot \sqrt{n}$$

where if n is even, then $1 \leq i \leq \dfrac{n}{2}$; if n is odd, then $1 \leq i \leq \dfrac{n-1}{2}$.

2   There are n points ($n \geq 4$) on a sphere with radius R, and not all of them lie on the same semi-sphere. Prove that among all the angles formed by any two of the n points and the sphere centre O (O is the vertex of the angle), there is at least one that is not less than $2 \arcsin \dfrac{\sqrt{6}}{3}$.

3   The positive integers α, β, γ are the roots of a polynomial $f(x)$ with degree 4 and the coefficient of the first term is 1. If there exists an integer such that $f(-1) = f^2(s)$.

Prove that $\alpha\beta$ is not a perfect square.

–   Quiz 2

1   Let $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$, with $n \geq 2$, be a real-coefficient polynomial. Prove that if there exists $a > 0$ such that

$$P_n(x) = (x+a)^2 \left(\sum_{i=0}^{n-2} b_i x^i \right),$$

where b_i are positive real numbers, then there exists some i, with $1 \leq i \leq n - 1$, such that

$$a_i^2 - 4a_{i-1}a_{i+1} \leq 0.$$

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2 A_1, B_1 and C_1 are the projections of the vertices A, B and C of triangle ABC on the respective sides. If $AB = c$, $AC = b$, $BC = a$ and $AC_1 = 2tAB$, $BA_1 = 2rBC$, $CB_1 = 2\mu AC$. Prove that:

$$\frac{a^2}{b^2} \cdot \left(\frac{t}{1-2t}\right)^2 + \frac{b^2}{c^2} \cdot \left(\frac{r}{1-2r}\right)^2 + \frac{c^2}{a^2} \cdot \left(\frac{\mu}{1-2\mu}\right)^2 + 16tr\mu \geq 1$$

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3   Let $p_i \geq 2$, $i = 1, 2, \cdots n$ be n integers such that any two of them are relatively prime. Let:

$$P = \{x = \sum_{i=1}^{n} x_i \prod_{j=1, j \neq i}^{n} p_j \mid x_i \text{is a non - negative integer}, i = 1, 2, \cdots n\}$$

Prove that the biggest integer M such that $M \notin P$ is greater than $\dfrac{n-2}{2} \cdot \prod_{i=1}^{n} p_i$, and also find M.

–   Quiz 3

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1   Given triangle ABC and $AB = c$, $AC = b$ and $BC = a$ satisfying $a \geq b \geq c$, BE and CF are two interior angle bisectors. P is a point inside triangle AEF. R and Q are the projections of P on sides AB and AC.

Prove that $PR + PQ + RQ < b$.

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2   For any two rational numbers p and q in the interval $(0, 1)$ and function f, there is always $f\left(\dfrac{p+q}{2}\right) \leq \dfrac{f(p) + f(q)}{2}$. Then prove that for any rational numbers $\lambda, x_1, x_2 \in (0, 1)$, there is always:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_i) + (1 - \lambda)f(x_2)$$

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3   Sequence $\{f_n(a)\}$ satisfies $f_{n+1}(a) = 2 - \dfrac{a}{f_n(a)}$, $f_1(a) = 2$, $n = 1, 2, \cdots$. If there exists a natural number n, such that $f_{n+k}(a) = f_k(a), k = 1, 2, \cdots$, then we call the non-zero real a a **periodic point** of $f_n(a)$.

Prove that the sufficient and necessary condition for a being a **periodic point** of $f_n(a)$ is $p_n(a - 1) = 0$, where $p_n(x) = \sum\limits_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k C_n^{2k+1} x^k$, here we define $\frac{a}{0} = \infty$ and $\frac{a}{\infty} = 0$.

–   Quiz 4

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1   Given a positive integer n, for all positive integers $a_1, a_2, \cdots, a_n$ that satisfy $a_1 = 1, a_{i+1} \leq a_i + 1$, find $\sum\limits_{i=1}^{n} a_1 a_2 \cdots a_i$.

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2   Circles ω_1 and ω_2 intersect at points A and B. Points C and D are on circles ω_1 and ω_2, respectively,
such that lines AC and AD are tangent to circles ω_2 and ω_1, respectively.
Let I_1 and I_2 be the incenters of triangles ABC and ABD, respectively. Segments $I_1 I_2$ and AB intersect at E.
Prove that: $\frac{1}{AE} = \frac{1}{AC} + \frac{1}{AD}$

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3   Find all groups of positive integers (a, x, y, n, m) that satisfy $a(x^n - x^m) = (ax^m - 4)y^2$ and $m \equiv n \pmod 2$ and ax is odd.

–   Quiz 5

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1   In acute triangle ABC, show that:
$\sin^3 A \cos^2 (B - C) + \sin^3 B \cos^2 (C - A) + \sin^3 C \cos^2 (A - B) \leq 3 \sin A \sin B \sin C$

and find out when the equality holds.

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2 m and n are positive integers. In a 8×8 chessboard, (m, n) denotes the number of grids a Horse can jump in a chessboard (m horizontal n vertical or n horizontal m vertical). If a (m, n)**Horse** starts from one grid, passes every grid once and only once, then we call this kind of Horse jump route a **H Route**. For example, the $(1, 2)$**Horse** has its **H Route**. Find the smallest positive integer t, such that from any grid of the chessboard, the $(t, t + 1)$**Horse** does not has any **H Route**.

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3 n sets $S_1, S_2 \cdots, S_n$ consists of non-negative numbers. x_i is the sum of all elements of S_i, prove that there is a natural number k, $1 < k < n$, and:

$$\sum_{i=1}^{n} x_i < \frac{1}{k+1}\left[k \cdot \frac{n(n+1)(2n+1)}{6} - (k+1)^2 \cdot \frac{n(n+1)}{2}\right]$$

and there exists subscripts i, j, t, and l (at least 3 of them are distinct) such that $x_i + x_j = x_t + x_l$.

–   Quiz 6

1   Given that $a_1 = 1$, $a_2 = 5$, $a_{n+1} = \dfrac{a_n \cdot a_{n-1}}{\sqrt{a_n^2 + a_{n-1}^2 + 1}}$. Find a expression of the general term of

       $\{a_n\}$.

2 $\odot O_1$ and $\odot O_2$ meet at points P and Q. The circle through P, O_1 and O_2 meets $\odot O_1$ and $\odot O_2$ at points A and B. Prove that the distance from Q to the lines PA, PB and AB are equal.

       (Prove the following three cases: O_1 and O_2 are in the common space of $\odot O_1$ and $\odot O_2$; O_1 and O_2 are out of the common space of $\odot O_1$ and $\odot O_2$; O_1 is in the common space of $\odot O_1$ and $\odot O_2$, O_2 is out of the common space of $\odot O_1$ and $\odot O_2$.

3   There is a game. The magician let the participant think up a positive integer (at least two digits). For example, an integer $\overline{a_1 a_2 \cdots a_n}$ is rearranged as $\overline{a_{i_1} a_{i_2} \cdots a_{i_n}}$, that is, $i_1, i_2, \cdots, i_n$ is a permutation of $1, 2, \cdots, n$. Then we get $n! - 1$ integers. The participant is asked to calculate the sum of the $n! - 1$ numbers, then tell the magician the sum S. The magician claims to be able to know the original number when he is told the sum S. Try to decide whether the magician can be successful or not.

–   Quiz 7

1   Circle O is inscribed in a trapezoid $ABCD$, $\angle A$ and $\angle B$ are all acute angles. A line through O intersects AD at E and BC at F, and satisfies the following conditions:

       (1) $\angle DEF$ and $\angle CFE$ are acute angles.

       (2) $AE + BF = DE + CF$.

       Let $AB = a$, $BC = b$, $CD = c$, then use a, b, c to express AE.

2   Find all non-negative integers m and n, such that $(2^n - 1) \cdot (3^n - 1) = m^2$.

3   Given positive integer $m \geq 17$, $2m$ contestants participate in a circular competition. In each round, we devide the $2m$ contestants into m groups, and the two contestants in one group play against each other. The groups are re-divided in the next round. The contestants compete for $2m - 1$ rounds so that each contestant has played a game with all the $2m - 1$ players. Find the least possible positive integer n, so that there exists a valid competition and after n rounds, for any 4 contestants, non of them has played with the others or there have been at least 2 games played within those 4.

–   Quiz 8

1 A is a set of points on the plane, L is a line on the same plane. If L passes through one of the points in A, then we call that L passes through A.

(1) Prove that we can divide all the rational points into 100 pairwisely non-intersecting point sets with infinity elements. If for any line on the plane, there are two rational points on it, then it passes through all the 100 sets.

(2) Find the biggest integer r, so that if we divide all the rational points on the plane into 100 pairwisely non-intersecting point sets with infinity elements with any method, then there is at least one line that passes through r sets of the 100 point sets.

2   Does there exist 2002 distinct positive integers $k_1, k_2, \cdots k_{2002}$ such that for any positive integer $n \geq 2001$, one of $k_1 2^n + 1, k_2 2^n + 1, \cdots, k_{2002} 2^n + 1$ is prime?

3   For positive integers a, b, c let α, β, γ be pairwise distinct positive integers such that

$$\begin{cases} ca & = \alpha + \beta + \gamma, \\ b & = \alpha \cdot \beta + \beta \cdot \gamma + \gamma \cdot \alpha, \\ c^2 & = \alpha\beta\gamma. \end{cases}$$

Also, let λ be a real number that satisfies the condition

$$\lambda^4 - 2a\lambda^2 + 8c\lambda + a^2 - 4b = 0.$$

Prove that λ is an integer if and only if α, β, γ are all perfect squares.